

On the Minimum Coprime Number of Some Graphs

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Abstract

Let G be a finite simple graph. A coprime labeling of G is an injective assignment of positive integers to the vertices such that adjacent vertices receive relatively prime labels. The minimum coprime number of G , denoted by $\text{pr}(G)$, is the smallest integer k for which G admits a coprime labeling using labels from $\{1, 2, \dots, k\}$. In this paper, we determine the exact minimum coprime number of several graph families, namely the disjoint union of cycle graphs, the disjoint union of complete graphs, and dumbbell graphs. For the graph mC_n with n odd, explicit coprime labeling constructions are provided and exact values of $\text{pr}(mC_n)$ are obtained for several infinite families. We also establish exact minimum coprime numbers for mK_n and characterize prime and coprime labelings of dumbbell graphs according to the parity of their parameters. Furthermore, constructive labeling functions are presented for all considered graph classes. These results extend previous studies on minimum coprime labeling for graph operations and disconnected graph families.

Keywords

Coprime Labeling, Disjoint Union, Dumbbell Graph, Minimum Coprime Number

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1. INTRODUCTION

Graph labeling is a function that assigns integers to the elements of a graph according to certain prescribed conditions (Gallian, 2024). Since its introduction, graph labeling has become one of the most active areas of graph theory due to its rich mathematical structure and broad range of applications. Practical applications of graph labeling include radio frequency assignment (Griggs and Yeh, 1992), coding theory (Aigner and Triesch, 1994), channel numbering in data transmission systems (Jin and Yeh, 2005), communication networks (Prasanna et al., 2014), computer science applications (Vinutha and Arathi, 2017), and message encoding (Prihandoko et al., 2019), among others. In general, graph labeling can be classified into three main categories: vertex labeling, edge labeling, and total labeling (Wijayanti et al., 2023).

Among the various graph labeling schemes, prime labeling has attracted considerable attention because of its close connection to number theory. Let $G = (V(G), E(G))$ be a simple graph of order $n = |V(G)|$, where $V(G)$ and $E(G)$ represent the vertex set and edge set of G , respectively. The graph G is said to admit a prime labeling if there exists a bijection $\xi : V(G) \rightarrow \{1, 2, \dots, n\}$ such that $\gcd(\xi(u), \xi(v)) = 1$ for every edge $uv \in E(G)$. This concept was introduced by En-

tringer and later popularized by Tout et al. (1982). Since then, prime labeling has inspired numerous variations, including k -prime labeling (Vaidya and Prajapati, 2011), neighborhood-prime labeling (Patel and Shrimali, 2015), SD-prime cordial labeling (Lau et al., 2016), total prime labeling (Prajapati and Gajjar, 2017), odd prime labeling (Prajapati and Shah, 2018), and coprime labeling (Berliner et al., 2016).

From a number-theoretic perspective, coprime labeling is closely related to the distribution of prime numbers and relatively prime integer pairs. A classical result states that the probability that two randomly chosen positive integers are relatively prime is equal to $\frac{6}{\pi^2}$ (Chidambaraswamy and Sitaramachandrarao, 1987). Consequently, coprimality naturally arises in graph labeling problems where adjacency imposes arithmetic constraints on the assigned labels. Determining the smallest label set that satisfies these coprimality conditions leads to an optimization problem that connects graph structure with number-theoretic properties. Therefore, the study of coprime labeling not only enriches graph labeling theory but also provides insight into how arithmetic properties interact with graph invariants such as degree and connectivity.

Since not every graph admits a prime labeling, Berliner et al. (2016) introduced the concept of coprime labeling as a natural

generalization of prime labeling. Let $G = (V(G), E(G))$ be a graph of order n . A coprime labeling of G is an injective function $\xi : V(G) \rightarrow \{1, 2, \dots, k\}$ such that $\gcd(\xi(u), \xi(v)) = 1$ for every edge $uv \in E(G)$. The smallest integer k for which G admits a coprime labeling is called the minimum coprime number of G , denoted by $\text{pr}(G)$. When $k = n$, a coprime labeling coincides with a prime labeling. Therefore, the minimum coprime number can be viewed as a measure of how far a graph is from admitting a prime labeling.

Research on prime and coprime labeling has developed rapidly since their introduction. Numerous families of graphs have been shown to admit prime labelings, beginning with paths, stars, caterpillars with maximum degree five, and cycles (Tout et al., 1982). Later, Lee (2020) established prime labelings for $\text{Amal}(G, v_0, m)$, where G is a cycle, a path, or an even wheel. Deretsky et al. (1991) proved that unions of even cycles (C_{2k}) admit prime labelings, while Seoud et al. (1998) extended these results to fan, helm, and flower graphs. Subsequently, Seoud and Youssef (1999) investigated book graphs, and Rao (2002) demonstrated prime labelings for spider and olive tree graphs. Youssef and Elsakhawi (2007) established prime labelings for unions of cycle and star graphs, whereas Pikhurko (2007) showed that all trees of order at most 50 admit prime labelings. Robertson and Small (2009) further obtained prime labelings for palm tree graphs, banana tree graphs, binomial graphs, and several families of spider graphs. Additional results have been reported for generalized Petersen graphs (Prajapati and Gajjar, 2015), bistar graphs (Ashokkumar and Maragathavalli, 2015), ladder graphs (Ghorbani and Kamali, 2016), brush graphs (Samuel and Kalaivani, 2018), H-graphs (Ganesan et al., 2019), torch graphs (Wilson and Jini, 2021), volcano graphs (Komarullah et al., 2026b), and broom graphs (Komarullah et al., 2026a).

The concept of the minimum coprime number has also received increasing attention. Berliner et al. (2016) initiated this line of research by determining the minimum coprime numbers of complete bipartite graphs. Subsequently, Asplund and Fox (2017) investigated the minimum coprime numbers of complete graphs, wheel graphs, and disjoint unions of odd cycles. Later, Asplund and Fox (2021) extended these results to prism graphs and generalized Petersen graphs. Lee (2020) studied the minimum coprime numbers of several graph classes, including $(P_m + P_n)$ and $(C_m + C_n)$. More recently, Komarullah et al. (2022) determined the minimum coprime numbers of amalgamated wheel graphs, while Komarullah et al. (2024) established coprime labeling results for amalgamations of complete graphs and diamond graphs.

Although previous studies have examined coprime labeling on several graph classes and operations, the investigation of the disjoint union of multiple graphs remains limited. In particular, the work of Asplund and Fox (2017) only considered the disjoint union of two odd cycles. Motivated by this limitation, this study extends the analysis to the graph mC_n , which represents the disjoint union of m copies of a cycle graph of order n . By examining this more general structure, this paper provides a

broader understanding of the minimum coprime number for families of graphs formed from multiple cycles. Furthermore, this study also investigates the minimum coprime number of the graph mK_n , which is the disjoint union of m complete graphs. The selection of this graph is motivated by previous studies on coprime labeling of complete graphs (Asplund and Fox, 2017) and related graph constructions (Komarullah et al., 2024). While earlier research has examined coprime labeling on single complete graphs and on amalgamated complete graphs, the behavior of coprime labeling on the disjoint union of multiple complete graphs has not been widely explored. In addition, this paper explores the existence of coprime labeling on graphs obtained through the comb operation applied to these graph families. This operation generates more complex graph structures, and therefore it is important to investigate whether the labeling patterns constructed on the base graphs can still be preserved after the operation is applied.

Finally, this study also analyzes coprime and prime labeling on dumbbell graphs, which consist of two cycle graphs connected by a path. The dumbbell graph is particularly interesting because its structural configuration may admit either prime labeling or only coprime labeling depending on the parity of its parameters. By examining these conditions, this paper determines the minimum coprime number of dumbbell graphs and characterizes the situations in which the graph admits prime labeling. Therefore, the contributions of this paper include determining the minimum coprime number of the graph mC_n as a generalization of previous results on disjoint unions of cycles, determining the minimum coprime number of the graph mK_n as an extension of studies on complete graphs and their related constructions, investigating coprime labeling on graphs obtained through the comb operation, and characterizing coprime and prime labeling on dumbbell graphs under various parameter conditions.

2. EXPERIMENTAL SECTION

This study employs a constructive graph-labeling approach combined with number-theoretic arguments to determine exact minimum coprime numbers of several graph families, namely mC_n , mK_n , and dumbbell graphs.

The proof strategy used throughout the paper consists of two complementary parts.

- First, lower bounds for the minimum coprime number are established by analyzing parity distributions, adjacency restrictions, and structural properties of the considered graphs. Since two adjacent even integers cannot be relatively prime, counting arguments involving the number of odd and even labels are used to derive necessary conditions for the existence of coprime labelings.
- Second, explicit labeling functions are constructed to attain these lower bounds. For each graph family, a deterministic labeling function $\xi : V(G) \rightarrow \{1, 2, \dots, k\}$ is proposed, where k denotes the candidate value of $\text{pr}(G)$. The constructions are designed according to the graph structure and rely on arithmetic properties of consecutive

integers, parity patterns, and divisibility considerations. To verify that the proposed labeling is a coprime labeling, the following steps are performed:

- (i) proving that the labeling function is injective;
- (ii) verifying that every pair of adjacent vertices receives relatively prime labels;
- (iii) establishing lower bounds using parity and counting arguments;
- (iv) showing that the constructed labeling attains the lower bound.

Whenever the lower and upper bounds coincide, exact values of the minimum coprime number are obtained. For prime labeling results, it is additionally shown that the constructed mapping is bijective, implying that the labels constitute a permutation of $\{1, 2, \dots, |V(G)|\}$.

The labeling constructions developed in this paper are deterministic and formula-based. Consequently, assigning labels to all vertices requires $O(|V(G)|)$ arithmetic operations, whereas verification of the coprimality condition requires $O(|E(G)|)$ greatest common divisor computations. Therefore, the overall construction and verification process can be performed in $O(|V(G)| + |E(G)|)$ time.

The following definitions and auxiliary result will be used throughout the paper.

Definition 2.1. Dumbbell graph $D_{m,n,q}$ consists of two cycle graphs (C_m and C_n) and one path graph (P_{q+2}). A dumbbell graph $D_{m,n,q}$ is constructed by attaching one vertex from each of the cycle graphs C_m and C_n to the end vertex of the path graph P_{q+2} (Wang et al., 2010).

Definition 2.2. Let G and H be connected graphs and $u \in V(H)$. The comb operation of graphs G and H is defined as a graph denoted by $G \triangleright H$, obtained by taking one copy of G and $|G|$ copy of H , then attaching vertex u from each copy of graph H to vertex- i^{th} of graph G (Hora and Obata, 2007).

Lemma 2.3. Let α and β be positive integers.

- (a) If α and β are consecutive, then $\gcd(\alpha, \beta) = 1$ (Sukirman, 2016).
- (b) If α is odd, then there exists a positive integer γ , which has no odd factors other than one, such that $\gcd(\alpha, \alpha + \gamma) = 1$ (Komarullah et al., 2022).

Definition 2.4. A mapping $\xi : A \rightarrow B$ is said to be injective (or one-to-one) if $f(x) = f(y)$ implies $x = y$ for all $x, y \in A$. Equivalently, distinct elements of A are assigned distinct images in B (Howson, 1972).

Definition 2.5. A mapping $\xi : A \rightarrow B$ is said to be bijective if it is both injective and surjective. That is, every element of B is the image of exactly one element of A (Howson, 1972).

3. RESULT AND DISCUSSION

In this section, we determine the minimum coprime number of the graphs mC_n, mK_n , and the dumbbell graph.

3.1 Coprime Labeling of the Graph mC_n

The graph mC_n is the disjoint union of m cycle graphs of order n . The vertex set of mC_n is defined as $V(mC_n) = \{v_b^a \mid a = 1, 2, \dots, m; b = 1, 2, \dots, n\}$, and the edge set is defined as $E(mC_n) = \{v_1^a v_n^a \mid a = 1, 2, \dots, m\} \cup \{v_b^a v_{b+1}^a \mid a = 1, 2, \dots, m; b = 1, 2, \dots, n-1\}$. Figure 1 illustrates the graph mC_n , including the notation for its vertices and edges.

Lemma 3.1. For every $m \geq 3$ and n is odd, $\text{pr}(mC_n) \geq mn + m - 1$.

Proof. Since the graph mC_n consists of mn vertices, any prime labeling of an odd cycle C_n requires exactly $\frac{n+1}{2}$ odd labels and $\frac{n-1}{2}$ even labels. This follows from the fact that two adjacent even labels cannot be relatively prime. Consequently, each cycle C_n must contain at least $\frac{n+1}{2}$ odd labels. Therefore, the disjoint union mC_n requires $\frac{m(n+1)}{2}$ odd labels and $\frac{m(n-1)}{2}$ even labels. However, the set $\{1, 2, \dots, mn\}$ does not contain a sufficient number of odd labels. Since there are $\frac{m(n+1)}{2}$ odd labels in the set $\{1, 2, \dots, mn + m - 1\}$, it follows that $\text{pr}(mC_n) \geq mn + m - 1$.

Theorem 3.2. For every $m \geq 3$ and $n = 2^k + 1$ with $k \in \mathbb{Z}$ and $k \geq 1$, $\text{pr}(mC_n) = mn + m - 1$.

Proof. Based on Lemma 3.1, we obtain $\text{pr}(mC_n) \geq mn + m - 1$. Next, we show that $\text{pr}(mC_n) \leq mn + m - 1$ for $n = 2^k + 1$ with $k \geq 1$ by constructing a coprime labeling of the graph mC_n . For each $a = 1, 2, \dots, m$, define the injection $\xi : V(mC_n) \rightarrow \{1, 2, \dots, mn + m - 1\}$ as follows.

$$\xi(v_b^a) = (n+1)(a-1) + b, \quad b = 1, 2, \dots, n.$$

For $b = 1, 2, \dots, n-1$, the labels $\xi(v_b^a)$ and $\xi(v_{b+1}^a)$ are consecutive integers. Hence, based on Lemma 2.3 (a), $\gcd(\xi(v_b^a), \xi(v_{b+1}^a)) = 1$. Furthermore, $|\xi(v_n^a) - \xi(v_1^a)| = |((n+1)(a-1) + n) - ((n+1)(a-1) + 1)| = n - 1$. Since $n = 2^k + 1$, we obtain $|\xi(v_n^a) - \xi(v_1^a)| = 2^k$. Because 2^k has no odd divisors other than 1, by Lemma 2.3 (b) we have $\gcd(\xi(v_n^a), \xi(v_1^a)) = 1$. Therefore, $\text{pr}(mC_n) \leq mn + m - 1$ for $n = 2^k + 1$. Together with the lower bound, we conclude that $\text{pr}(mC_n) = mn + m - 1$ for $n = 2^k + 1$.

Figure 2 illustrates a coprime labeling of the graph $3C_5$.

The following corollaries are direct consequences of Theorem 3.2 and describe the minimum coprime number of the graph mC_n for some special values of m and n .

Corollary 3.3. For every $m \geq 3$ and $n = 2^k + 1$ with $k \geq 1$, $\text{pr}(P_m \triangleright C_n) = mn + m - 1$.

Proof. Let $a = 1, 2, \dots, m-1$. Consider $\gcd(\xi(v_1^a), \xi(v_1^{a+1})) = \gcd(((n+1)(a-1) + 1), ((n+1)a + 1)) = \varphi$. Then there exist integers x and y such that $((n+1)(a-1) + 1) = \varphi x$ and $((n+1)a + 1) = \varphi y$. Subtracting the first equation from the second gives $\varphi(y - x) = n + 1$. Therefore, φ is a divisor of $n + 1$. Let τ_i be any odd divisor of $n + 1$ with $\tau_i > 1$. By

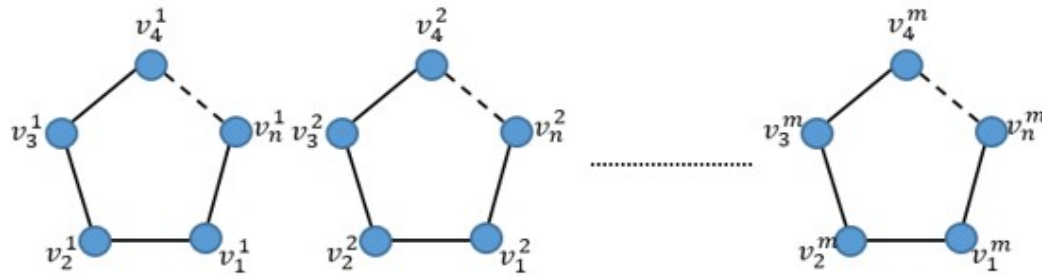


Figure 1. The Graph mC_n

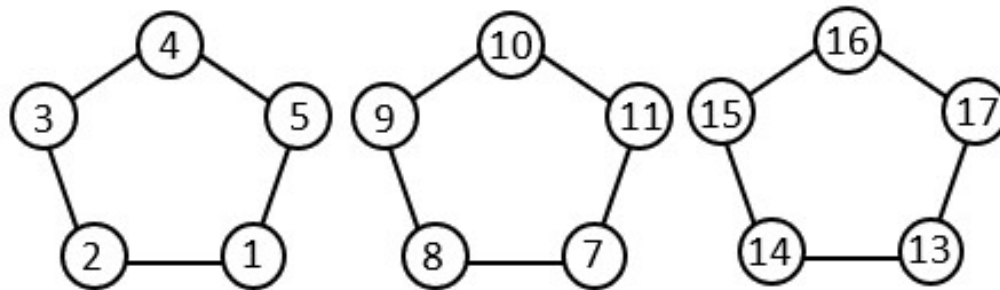


Figure 2. Coprime Labeling of the Graph $3C_5$

the labeling construction, neither $\xi(v_1^a)$ nor $\xi(v_1^{a+1})$ is divisible by τ_i , that is, $\xi(v_1^a) \not\equiv 0 \pmod{\tau_i}$ and $\xi(v_1^{a+1}) \not\equiv 0 \pmod{\tau_i}$. Hence, τ_i cannot be a common divisor of $\xi(v_1^a)$ and $\xi(v_1^{a+1})$, implying that $\varphi \neq \tau_i$. Moreover, both $\xi(v_1^a)$ and $\xi(v_1^{a+1})$ are odd integers, and therefore any common divisor φ must also be odd. Since every odd divisor of $n + 1$ greater than 1 is excluded, the only possible value of φ is $\varphi = 1$. Consequently, $\gcd(\xi(v_1^a), \xi(v_1^{a+1})) = 1$. Therefore, joining the vertices v_1^a and v_1^{a+1} for $a = 1, 2, \dots, m - 1$ in the graph mC_n produces the graph $P_m \triangleright C_n$. Since every newly added edge also joins vertices with relatively prime labels, the labeling remains a coprime labeling. Hence, $\text{pr}(P_m \triangleright C_n) = mn + m - 1$.

Corollary 3.4. For every $m \geq 3$ and $n = 2^k + 1$ with $k \geq 1$, $\text{pr}(C_m \triangleright C_n) = mn + m - 1$.

Proof. If the vertices v_1^1 and v_1^m in the graph $P_m \triangleright C_n$ are adjacent, the resulting graph is $C_m \triangleright C_n$. Since $\xi(v_1^1) = 1$ is relatively prime to every integer, the coprime labeling of $P_m \triangleright C_n$ remains valid for $C_m \triangleright C_n$.

Figure 3 illustrates examples of coprime labelings for the graphs $P_3 \triangleright C_5$ and $C_3 \triangleright C_5$.

In order to support the main results, we first present several preliminary results related to the structure of the graph mC_n . These propositions are introduced because a general result for the coprime labeling pattern of mC_n when n is odd is not yet available.

Proposition 3.5. For every $m \geq 3$, $\text{pr}(mC_7) = 8m - 1$.

Proof. Based on Lemma 3.1, we obtain that $\text{pr}(mC_7) \geq 8m - 1$. Next, we show that $\text{pr}(mC_7) \leq 8m - 1$ by constructing a coprime labeling of the graph mC_7 . Define the injective function $\xi: V(mC_7) \rightarrow \{1, 2, \dots, 8m - 1\}$, which is divided into two cases as follows.

(i) Case $a \not\equiv 2 \pmod{3}$

$$\xi(v_b^a) = 8(a-1) + b, \quad a = 1, 2, \dots, m \text{ and } b = 1, 2, \dots, n.$$

The vertex labels of v_b^a and v_{b+1}^a for $b = 1, 2, \dots, 6$ are relatively prime since they are consecutive integers. Next, consider $\gcd(\xi(v_7^a), \xi(v_1^a)) = \gcd(8a - 1, 8a - 7) = k$. Since $8a - 1$ and $8a - 7$ are odd integers and are not congruent to 0 (mod 3), it follows that k must be odd and $k \neq 3$. Suppose that $8a - 1 = kx$ and $8a - 7 = ky$, which implies $k(x - y) = 6$. Since k is odd and $k \neq 3$, we obtain $k = 1$. Hence, $\gcd(\xi(v_7^a), \xi(v_1^a)) = 1$.

(ii) Case $a \equiv 2 \pmod{3}$

$$\xi(v_b^a) = \begin{cases} 8(a-1) + b - 1, & a \equiv 2 \pmod{15} \text{ and } b = 1, 2, \\ 8(a-1) - b + 3, & a \not\equiv 2 \pmod{15} \text{ and } b = 1, 2, \\ 8(a-1) + b, & a = 1, 2, \dots, m \text{ and } b = 3, 4, \dots, n. \end{cases}$$

For the second case, the adjacency relations are examined individually. Consecutive labels are relatively prime by Lemma 2.3 (a), whereas nonconsecutive pairs differ by an even integer and both labels remain odd. Since none of the labels is divisible by the corresponding odd divisor, their greatest common divisor is equal to one. Therefore, every adjacent pair satisfies the coprimality condition.

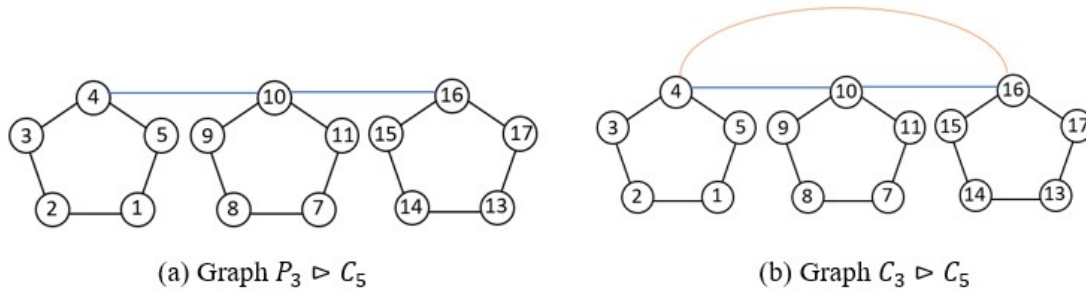


Figure 3. Coprime Labelings of the Graphs $P_3 \succ C_5$ and $C_3 \succ C_5$

Based on the above construction, every pair of adjacent vertices receives relatively prime labels. Hence, $\text{pr}(mC_7) \leq 8m - 1$. Combining this result with the previously established lower bound $\text{pr}(mC_7) \geq 8m - 1$, we conclude that $\text{pr}(mC_7) = 8m - 1$.

In a similar approach to Proposition 3.5, we obtain the following corollaries.

Corollary 3.6. For every $m \geq 3$, $\text{pr}(P_m \succ C_7) = 8m - 1$.

Proof. Since $\text{gcd}(\xi(v_1^a), \xi(v_1^{a+1})) = 1$ for all $a = 1, 2, \dots, m - 1$, the labeling construction given in Proposition 3.5 remains valid for the graph $P_m \succ C_7$. Therefore, the labeling satisfies the coprime condition on every edge, implying that $\text{pr}(P_m \succ C_7) = 8m - 1$.

Corollary 3.7. For every $m \geq 3$, $\text{pr}(C_m \succ C_7) = 8m - 1$.

Proof. Since $\xi(v_1^1) = 1$ in the labeling construction of Corollary 3.6 and 1 is relatively prime to every positive integer, the same labeling pattern can be applied to the graph $C_m \succ C_7$. Consequently, the labeling remains a coprime labeling, implying that $\text{pr}(C_m \succ C_7) = 8m - 1$.

Proposition 3.8. For every $m \geq 3$, $\text{pr}(mC_{11}) = 12m - 1$.

Proof. By applying the same argument as in Lemma 3.1, we obtain $\text{pr}(mC_{11}) \geq 12m - 1$. To establish the upper bound, we construct a coprime labeling of the graph mC_{11} showing that $\text{pr}(mC_{11}) \leq 12m - 1$. Define injective function $\xi: V(mC_{11}) \rightarrow \{1, 2, \dots, 12m - 1\}$ as follows.

For $a = 1, 2, \dots, m$

$$\xi(v_b^a) = \begin{cases} 12(a-1) + b, & b = 1, 2, \dots, 9, \\ 12(a-1) - b + 21, & b = 10, 11. \end{cases}$$

It is known that $|\xi(v_9^a) - \xi(v_{10}^a)| = |(12a-3) - (12a-1)| = 2$. Since 2 has no odd divisors other than 1, it follows from Lemma 2.3 (b) that $\text{gcd}(\xi(v_9^a), \xi(v_{10}^a)) = 1$. Next, consider $\text{gcd}(\xi(v_{11}^a), \xi(v_1^a)) = \text{gcd}(12a-11, 12a-2) = k$. Since $12a-11$ is odd, k must also be odd. Moreover, $\xi(v_{11}^a)$ and $\xi(v_1^a)$ are not congruent to 0 (mod 3) and 0 (mod 9), implying that $k \neq 3$ and $k \neq 9$. Suppose that $12a-11 = kx$ and $12a-2 = ky$, which yields $k(y-x) = 9$. Since k is odd and $k \neq \{3, 9\}$,

we obtain $k = 1$. Hence, $\text{gcd}(\xi(v_{11}^a), \xi(v_1^a)) = 1$. Therefore, $\text{pr}(mC_{11}) \leq 12m - 1$. Together with the lower bound, we conclude that $\text{pr}(mC_{11}) = 12m - 1$.

By applying an approach similar to that used in Proposition 3.8, the following corollaries can be established.

Corollary 3.9. For every $m \geq 3$, $\text{pr}(P_m \succ C_{11}) = 12m - 1$.

Proof. Observe that, in Proposition 3.8, $\text{gcd}(\xi(v_1^a), \xi(v_1^{a+1})) = \text{gcd}(12a-11, 12a+1) = \varphi$. Then there exist integers x and y such that $12a-11 = \varphi x$ and $12a+1 = \varphi y$. Subtracting the first equation from the second yields $\varphi(y-x) = 12$. Hence, φ is a divisor of 12. Since both $\xi(v_1^a)$ and $\xi(v_1^{a+1})$ are odd integers and neither is divisible by 3, any common divisor φ must be odd and satisfy $\varphi \neq 3$. The only odd divisors of 12 are 1 and 3; therefore, the only possible value of φ is $\varphi = 1$. Consequently, $\text{gcd}(\xi(v_1^a), \xi(v_1^{a+1})) = 1$ for all $a = 1, 2, \dots, m-1$. Therefore, the labeling construction in Proposition 3.8 can be extended to the graph $P_m \succ C_{11}$, and the resulting labeling remains a coprime labeling. Hence, $\text{pr}(P_m \succ C_{11}) = 12m - 1$.

Corollary 3.10. For every $m \geq 3$, $\text{pr}(C_m \succ C_{11}) = 12m - 1$.

Proof. Observe that $\xi(v_1^1) = 1$ in the labeling construction of Corollary 3.9. Since 1 is relatively prime to every positive integer, connecting the vertices v_1^1 and v_1^m creates the graph $C_m \succ C_{11}$ without violating the coprime condition. Hence, the same labeling construction yields a coprime labeling of $C_m \succ C_{11}$, and therefore $\text{pr}(C_m \succ C_{11}) = 12m - 1$.

For a graph mC_n with n is even, the graph requires $\frac{mn}{2}$ odd labels and $\frac{mn}{2}$ even labels. The label set $\{1, 2, \dots, mn\}$ contains exactly $\frac{mn}{2}$ odd labels and $\frac{mn}{2}$ even labels, which makes it possible for the labels of every pair of adjacent vertices in the graph mC_n to be relatively prime. However, in this paper, we have not yet identified a general prime labeling for the graph mC_n when n is even. Nevertheless, the labeling patterns used in the coprime labeling framework suggest that prime labeling may exist for certain values of m . Therefore, in addition to the coprime labeling results obtained in this section, we present several examples of prime labelings for the disjoint union of even cycles for specific values of m , as stated in Proposition 3.11 and Proposition 3.12.

Proposition 3.11. For every even integer $n \geq 4$, the graph $2C_n$ is prime.

Proof. To prove that the graph $2C_n$ with even $n \geq 4$ is prime, we construct a bijective function $\xi: V(2C_n) \rightarrow \{1, 2, \dots, 2n\}$ and show that it satisfies the conditions of a prime labeling.

$$\xi(v_b^a) = \begin{cases} b+1, & a=1 \text{ and } b=1, 2, \dots, n, \\ n+b+1, & a=2 \text{ and } b=1, 2, \dots, n-1, \\ 1, & a=2 \text{ and } b=n. \end{cases}$$

Observe that $\xi(v_b^1)$ and $\xi(v_{b+1}^1)$ are consecutive integers for $b=1, 2, \dots, n-1$. Hence, by Lemma 2.3 (a), $\gcd(\xi(v_b^1), \xi(v_{b+1}^1)) = 1$. Note that $\xi(v_1^1) = 2$ and $\xi(v_n^1) = n+1$. Since $n+1$ is odd, it follows that $\gcd(\xi(v_1^1), \xi(v_n^1)) = 1$, because 2 is relatively prime to every odd integer. Similarly, for $b=1, 2, \dots, n-2$, the values $\xi(v_b^2)$ and $\xi(v_{b+1}^2)$ are consecutive integers. Thus, $\gcd(\xi(v_b^2), \xi(v_{b+1}^2)) = 1$. Furthermore, $\xi(v_n^2) = 1$, which is relatively prime to both $\xi(v_1^2)$ and $\xi(v_{n-1}^2)$. Therefore, the function ξ satisfies the conditions of a prime labeling. Hence, the graph $2C_n$, for even $n \geq 4$, is prime.

Proposition 3.12. For every even integer $n \geq 4$, the graph $3C_n$ is prime.

Proof. Similarly, define a bijection $\xi: V(3C_n) \rightarrow \{1, 2, \dots, 3n\}$ such that the conditions of a prime labeling are satisfied as follows.

(i) Case $n \equiv 1 \pmod{3}$

$$\xi(v_b^a) = \begin{cases} b+3, & a=1 \text{ and } b=1, 2, \dots, n, \\ b+1, & a=2 \text{ and } b=1, 2, \\ n+b+1, & a=2 \text{ and } b=3, 4, \dots, n, \\ 2n+b+1, & a=3 \text{ and } b=1, 2, \dots, n-1, \\ 1, & a=3 \text{ and } b=n. \end{cases}$$

For $a=1, 2$ and $b=1, 2, \dots, n-1$, the labels $\xi(v_b^a)$ and $\xi(v_{b+1}^a)$ are consecutive integers. Therefore, by Lemma 2.3 (a), $\gcd(\xi(v_b^a), \xi(v_{b+1}^a)) = 1$. Furthermore, $\xi(v_n^a)$ is odd for $a=1, 2$. Hence, $\gcd(\xi(v_1^1), \xi(v_n^1)) = \gcd(4, n+3) = 1$ and $\gcd(\xi(v_1^2), \xi(v_n^2)) = \gcd(2, 2n+1) = 1$. Next, for $a=3$ and $b=1, 2, \dots, n-2$, the labels $\xi(v_b^3)$ and $\xi(v_{b+1}^3)$ are also consecutive integers. Thus, by Lemma 2.3 (a), $\gcd(\xi(v_b^3), \xi(v_{b+1}^3)) = 1$. Moreover, $\xi(v_n^3) = 1$. Therefore, $\gcd(\xi(v_n^3), \xi(v_{n-1}^3)) = 1$ and $\gcd(\xi(v_n^3), \xi(v_1^3)) = 1$. Consequently, every pair of adjacent vertices receives relatively prime labels. Hence, the labeling function ξ satisfies the coprime labeling condition.

(ii) Case $n \not\equiv 1 \pmod{3}$

$$\xi(v_b^a) = \begin{cases} b+2, & a=1 \text{ and } b=1, 2, \dots, n, \\ n+b+2, & a=2 \text{ and } b=1, 2, \dots, n-1, \\ 2, & a=2 \text{ and } b=n, \\ 2n+b+1, & a=3 \text{ and } b=1, 2, \dots, n-1, \\ 1, & a=3 \text{ and } b=n. \end{cases}$$

For $a=1$, the labels assigned to the vertices v_b^1 and v_{b+1}^1 are relatively prime because they are consecutive integers. Furthermore, we have $\xi(v_1^1) \not\equiv 0 \pmod{3}$, and therefore $\gcd(\xi(v_1^1), \xi(v_n^1)) = \gcd(3, n+2) = 1$. For $a=2$, the labels assigned to the vertices v_b^2 and v_{b+1}^2 are also relatively prime since they are consecutive integers. In addition, $\xi(v_1^2)$ and $\xi(v_{n-1}^2)$ are odd integers, implying that $\gcd(\xi(v_n^2), \xi(v_1^2)) = \gcd(\xi(v_n^2), \xi(v_{n-1}^2)) = 1$, because $\xi(v_n^2) = 2$ is relatively prime to every odd integer. For $a=3$, the labels assigned to the vertices v_b^3 and v_{b+1}^3 are likewise relatively prime since they are consecutive integers. Finally, the label assigned to v_n^3 is equal to 1, which is relatively prime to every positive integer.

Based on the above construction, every pair of adjacent vertices receives relatively prime labels. Consequently, ξ defines a prime labeling of $3C_n$, and hence $3C_n$ is a prime graph for all even integers $n \geq 4$.

3.2 Coprime Labeling of the Graph mK_n

Let mK_n denote the disjoint union of m copies of the complete graph K_n . The vertex set and edge set are defined as follows.

$$V(mK_n) = \{v_a^b \mid a=1, 2, \dots, n; b=1, 2, \dots, m\}$$

$$E(mK_n) = \{v_a^b v_c^b \mid a \neq c; a, c=1, 2, \dots, n; b=1, 2, \dots, m\}$$

Figure 4 is an example of the notation of vertices and edges on the graph mK_n .

Before stating Theorem 3.13, we introduce the following notation. Let p_n denote the n^{th} prime number in the ordered sequence of prime numbers $(2, 3, 5, 7, 11, \dots)$. Thus, $p_1 = 2, p_2 = 3, p_3 = 5$, and so on. For example, p_3 represents the third prime number, namely 5.

Theorem 3.13. For every $n > 3$ and $m \geq 2$, $\text{pr}(mK_n) = p_{(n-1)m}$ where $p_{(n-1)m}$ denotes the $(n-1)m^{\text{th}}$ prime number.

Proof. We first show that $\text{pr}(mK_n) \geq p_{(n-1)m}$ for $n > 3$. Note that every vertex in the graph mK_n is adjacent to all other vertices. Thus, each complete graph requires one composite label and $n-1$ prime labels. Consequently, $\text{pr}(mK_n) \geq p_{(n-1)m}$, where $p_{(n-1)m}$ denotes the $(n-1)m^{\text{th}}$ prime number. Next, we show that $\text{pr}(mK_n) \leq p_{(n-1)m}$ by constructing a coprime labeling. Define the injection $\xi: V(mK_n) \rightarrow \{1, 2, \dots, p_{(n-1)m}\}$ as follows.

$$\xi(v_a^b) = \begin{cases} 1, & a=1 \text{ and } b=1, \\ 2b, & a=1 \text{ and } b=2, 3, \dots, m, \\ p_{(b-1)(n-1)+(a-1)}, & a=2, 3, \dots, n \text{ and } b=1, 2, \dots, m. \end{cases}$$

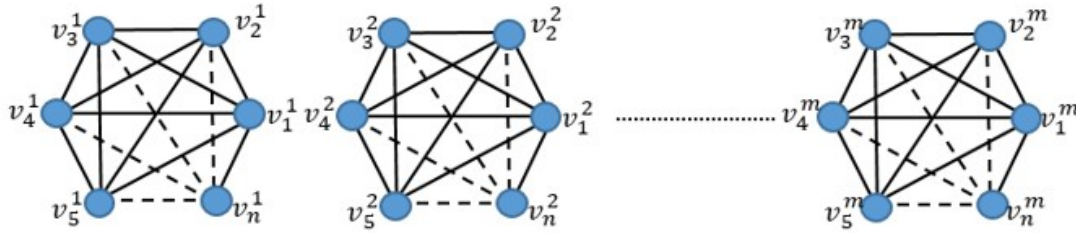


Figure 4. The Graph mK_n

Observe that all prime labels assigned in the construction are pairwise relatively prime. Moreover, each composite label assigned to v_1^b is selected so that none of its prime factors appears among the labels assigned to the vertices v_a^b , where $a = 2, 3, \dots, n$. Consequently, every composite label is relatively prime to each adjacent prime label. Therefore, the labeling satisfies the coprime condition on every edge of mK_n . Hence, $\text{pr}(mK_n) \leq p_{(n-1)m}$. Combining this with the lower bound yields $\text{pr}(mK_n) = p_{(n-1)m}$.

The following corollaries present immediate consequences of the previous theorem regarding the minimum coprime number of the graph mK_n . These results describe several special cases that can be directly derived from the general result.

Corollary 3.14. For every $n > 3$ and $m \geq 2$, $\text{pr}(P_m \triangleright K_n) = p_{(n-1)m}$ where $p_{(n-1)m}$ is the $(n-1)m^{\text{th}}$ prime number.

Proof. We know that the graph mK_n is the disjoint union of m copies of the complete graph K_n . If a vertex of the b^{th} graph K_n that is assigned a prime label is adjacent to a vertex of the $(b+1)^{\text{th}}$ graph K_n , then the graph $P_m \triangleright K_n$ is formed. Since both vertices are assigned prime labels, the greatest common divisor of the labels of every such pair of adjacent vertices is 1, that is, they are relatively prime. Therefore, the graph $P_m \triangleright K_n$ is a coprime graph where $\text{pr}(P_m \triangleright K_n) = p_{(n-1)m}$.

Corollary 3.15. For every $n > 3$ and $m \geq 2$, $\text{pr}(C_m \triangleright K_n) = p_{(n-1)m}$ where $p_{(n-1)m}$ is the $(n-1)m^{\text{th}}$ prime number.

Proof. Consider the graph $P_m \triangleright K_n$. If a vertex with a prime label in the first graph K_n is adjacent to a vertex with a prime label in the m^{th} graph K_n , then the graph $C_m \triangleright K_n$ is formed. Since both vertices are labeled with prime numbers, they are relatively prime. Therefore, the graph $C_m \triangleright K_n$ is a coprime graph where $\text{pr}(C_m \triangleright K_n) = p_{(n-1)m}$.

3.3 Coprime Labeling of Dumbbell Graphs

The dumbbell graph $D_{m,n,q}$ consists of the vertex set $V(D_{m,n,q}) = \{u_a \mid a = 1, 2, \dots, m\} \cup \{v_b \mid b = 1, 2, \dots, n\} \cup \{w_c \mid c = 1, 2, \dots, q\}$ and the edge set $E(D_{m,n,q}) = \{u_1 w_1, u_1 u_m, u_a u_{a+1} \mid a = 1, 2, \dots, m-1\} \cup \{w_q v_1, w_c w_{c+1} \mid c = 1, 2, \dots, q-1\} \cup \{v_1 v_n, v_b v_{b+1} \mid b = 1, 2, \dots, n-1\}$.

To elucidate the notation of vertices and edges on the dumbbell graph, an illustration is presented in Figure 5.

Theorem 3.16. For every m, n odd and q even, or m even and n, q odd, the dumbbell graph $D_{m,n,q}$ is a coprime graph with $\text{pr}(D_{m,n,q}) = m + n + q + 1$.

Proof. It has been shown that $\text{pr}(D_{m,n,q}) \geq m + n + q + 1$. Suppose that the dumbbell graph $D_{m,n,q}$, where m and n are odd and q is even, or m is even while n and q are odd, can be labeled using the set $\{1, 2, \dots, m + n + q\}$. It is well known that two even integers cannot be relatively prime. Therefore, in a coprime labeling, the labels of adjacent vertices must alternate between even and odd integers. The possible labeling patterns under these conditions are divided into the following two cases.

(i) Case m, n are odd and q is even

In order to determine the minimum coprime number, the labeling of the cycle C_m must start with an odd label. Since C_m has odd order, the final label of the cycle C_m is also odd. Consequently, the labeling of the vertices w_c , for $c = 1, 2, \dots, q$, can start with an even label because the vertex u_1 has an odd label. Since q is even, the label of the vertex w_q is odd. To obtain the minimum coprime number, the vertex v_1 is assigned an even label. Hence, the label of the vertex v_n is also even. Consequently, the labels of the vertices v_1 and v_n are not relatively prime. Therefore, the dumbbell graph $D_{m,n,q}$ with m and n odd and q even requires $\frac{m+q+n+2}{2}$ odd labels and $\frac{m+q+n-2}{2}$ even labels.

(ii) Case m is even and n, q are odd

Likewise, in this case, the cycle C_m starts with an odd label and ends with an even label. In order to obtain the minimum coprime number, the labeling of the vertices w_c , for $c = 1, 2, \dots, q$, can start with an odd label because the vertex u_n has an even label. Since q is odd, the label of the vertex w_q is also odd. To obtain the minimum coprime number, the vertex v_1 is assigned an even label. Hence, the label of the vertex v_n is also even. Consequently, the labels of the vertices v_1 and v_n are not relatively prime. Thus, the dumbbell graph $D_{m,n,q}$ with m even and n, q odd requires $\frac{m+q+n+2}{2}$ odd labels and $\frac{m+q+n-2}{2}$ even labels. The set $\{1, 2, \dots, m + n + q\}$ contains $\frac{m+q+n}{2}$ odd labels. Hence, one additional odd label is required, implying that $\text{pr}(D_{m,n,q}) \geq m + n + q + 1$.

Next, we show that $\text{pr}(D_{m,n,q}) \leq m + n + q + 1$ by constructing a coprime labeling of the graph $D_{m,n,q}$. The injective function

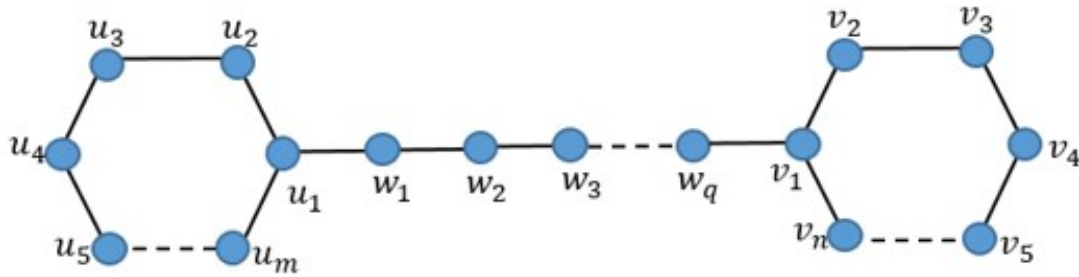


Figure 5. The Dumbbell Graph $D_{m,n,q}$

$\xi : V(D_{m,n,q}) \rightarrow \{1, 2, \dots, m+n+q+1\}$ is defined as follows and is divided into two cases.

(i) Case m, n are odd and q is even

$$\xi(u_a) = \begin{cases} a, & a = 1, 3, \\ 4, & a = 2, \\ a+1, & a = 4, 5, \dots, m. \end{cases}$$

$$\xi(v_b) = \begin{cases} 2, & b = 1, \\ m+q+b, & b = 2, 3, \dots, n-1, \\ m+q+n+1, & b = n. \end{cases}$$

$$\xi(w_c) = m+q+1-(c-1), \quad c = 1, 2, \dots, q.$$

The following shows that adjacent vertices have coprime labels.

- Observe that $\gcd(\xi(w_q), \xi(v_1)) = \gcd(m+2, 2) = 1$, since $m+2$ is odd and 2 is relatively prime to every odd integer.
- Since $m+q+2$ is odd, $\gcd(\xi(v_1), \xi(v_2)) = \gcd(2, m+q+2) = 1$.
- Note that $|\xi(v_{n-1}) - \xi(v_n)| = 2$. Since the integer 2 has no odd divisor other than 1, it follows from Lemma 3.3 (b) that $\gcd(\xi(v_{n-1}), \xi(v_n)) = 1$.
- Furthermore, $\gcd(\xi(v_1), \xi(v_n)) = \gcd(2, m+q+n+1) = 1$, since $m+q+n+1$ is odd and 2 is relatively prime to every odd integer.

(ii) Case m is even and n, q are odd

$$\xi(u_a) = \begin{cases} m+n+q+2, & a = 2, \\ a+1, & a = 1, 3, 4, \dots, m-1, \\ m+1, & a = m \text{ and } m \equiv 0 \pmod{3}, \\ 3, & a = m \text{ and } m \not\equiv 0 \pmod{3}. \end{cases}$$

$$\xi(v_b) = \begin{cases} m+q+b, & b = 1, 2, \dots, n-1, \\ 1, & b = n. \end{cases}$$

$$\xi(w_c) = \begin{cases} 3, & c = 1 \text{ and } m \equiv 0 \pmod{3}, \\ m+1, & c = 1 \text{ and } m \not\equiv 0 \pmod{3}, \\ m+c, & c = 2, 3, \dots, q. \end{cases}$$

Observe that $\xi(u_1) = 2$ is relatively prime to $\xi(w_1) = 3$. Furthermore, $\xi(w_q) = m+q$ is relatively prime to $\xi(v_1) =$

$m+q+1$ since they are consecutive integers. Moreover, the labels assigned to the vertices on each cycle and along the path are consecutive. This guarantees that every pair of adjacent vertices receives relatively prime labels.

Based on the above discussion, every pair of adjacent vertices receives relatively prime labels, thereby ensuring that $\text{pr}(D_{m,n,q}) \leq m+n+q+1$. Together with the lower bound $\text{pr}(D_{m,n,q}) \geq m+n+q+1$, it follows that $\text{pr}(D_{m,n,q}) = m+n+q+1$.

Theorem 3.17. The dumbbell $D_{m,n,q}$ is a prime graph under all conditions that are not covered by Theorem 3.16.

Proof. To show that the dumbbell $D_{m,n,q}$ is a prime graph under all conditions not covered by Theorem 3.16, we construct a function that satisfies the prime labeling rule for this graph. The bijective function $\xi : V(D_{m,n,q}) \rightarrow \{1, 2, \dots, m+n+q+1\}$ is defined in several cases as follows.

(i) The label of vertex v_b

- For $((m, n, \text{ and } q \text{ are odds}), (m, n, \text{ and } q \text{ are evens}), \text{ or } (m, q \text{ are odds, } n \text{ is even, and } m \equiv 1, 2 \pmod{3}))$

$$\xi(v_b) = \begin{cases} 1, & b = 1, \\ m+q+b, & b = 2, 3, \dots, n. \end{cases}$$

- For other conditions

$$\xi(v_b) = \begin{cases} m+q+b, & b = 1, 2, \dots, n-1, \\ 1, & b = n, \end{cases}$$

(ii) The label of vertex w_c

- For $((m, n, q \text{ are evens, and } m \equiv 0 \pmod{3}), (m, n \text{ are evens, } q \text{ is odd, and } m \equiv 0 \pmod{3}), \text{ or } (m, q \text{ are evens, } n \text{ is odd, and } m \equiv 0 \pmod{3}))$

$$\xi(w_c) = \begin{cases} 3, & c = 1, \\ m+c, & c = 2, 3, \dots, q. \end{cases}$$

- For other conditions

$$\xi(w_c) = m+c, \quad c = 1, 2, \dots, q.$$

(iii) The label of vertex u_a

a. For m, n , and q are odds

$$\xi(u_a) = \begin{cases} 4-a, & a = 1, 2, \text{ and } m \not\equiv 2 \pmod{3}, \\ m, & a = 1 \text{ and } m \equiv 2 \pmod{3}, \text{ or } a = m-1 \text{ and } m \equiv 0 \pmod{3}, \\ 2, & a = 2 \text{ and } m \equiv 2 \pmod{3}, \\ m+q+1, & a = 3, \\ a, & a = 4, 5, \dots, m-2 \text{ or } a = m-1, m \text{ and } m \equiv 1 \pmod{3}, \\ m-1, & a = m-1 \text{ and } m \equiv 2 \pmod{3}, \text{ or } a = m \text{ and } m \equiv 0 \pmod{3}, \\ 3, & a = m \text{ and } m \equiv 2 \pmod{3}. \end{cases}$$

b. For m, n , and q are evens

$$\xi(u_a) = \begin{cases} 4-a, & a = 1, 2, \text{ and } m \equiv 1 \pmod{3}, \\ m+1, & a = 1 \text{ and } m \equiv 0 \pmod{3}, \\ 2, & a = 2 \text{ and } m \equiv 0 \pmod{3}, \\ a+1, & a = 1, 3, 4, \dots, m-1 \text{ and } m \equiv 2 \pmod{3}, \\ m+q+1, & a = 3 \text{ and } m \not\equiv 2 \pmod{3}, \text{ or } a = 2 \text{ and } m \equiv 2 \pmod{3}, \\ a, & a = 4, 5, \dots, m \text{ and } m \not\equiv 2 \pmod{3}, \\ 3, & a = m \text{ and } m \equiv 2 \pmod{3}. \end{cases}$$

c. For m, n are evens and q is odd or m, q are evens and n is odd

$$\xi(u_a) = \begin{cases} a+1, & a = 1, 3, 4, \dots, m-1, \\ m+n+q, & a = 2, \\ m+1, & a = m \text{ and } m \equiv 0 \pmod{3}, \\ 3, & a = m \text{ and } m \not\equiv 0 \pmod{3}. \end{cases}$$

d. For m, q are odds and n is even

$$\xi(u_a) = \begin{cases} m, & a = 1 \text{ and } m \not\equiv 1 \pmod{3}, \\ 4-a, & a = 1 \text{ and } m \equiv 1 \pmod{3}, \\ 2, & a = 2, \\ m+q+1, & a = 3 \text{ and } m \equiv 0 \pmod{3}, \\ m+q+n, & a = 3 \text{ and } m \not\equiv 0 \pmod{3}, \\ a, & a = 4, 5, \dots, m-2 \text{ or } a = m-1, m \text{ and } m \equiv 1 \pmod{3}, \\ m-1, & a = m-1 \text{ and } m \equiv 2 \pmod{3} \text{ or } a = m \text{ and } m \equiv 0 \pmod{3}, \\ 3, & a = m \text{ and } m \equiv 2 \pmod{3} \text{ or } a = m-1 \text{ and } m \equiv 0 \pmod{3}. \end{cases}$$

e. For m is odd and n, q are evens

$$\xi(u_a) = \begin{cases} m, & a = 1 \text{ and } m \not\equiv 1 \pmod{3}, \\ 4-a, & a = 1 \text{ and } m \equiv 1 \pmod{3}, \\ 2, & a = 2, \\ m+q+n, & a = 3, \\ a, & a = 4, 5, \dots, m-1 \text{ or } a = m \text{ and } m \equiv 1 \pmod{3}, \\ 3, & a = m \text{ and } m \equiv 1 \pmod{3}. \end{cases}$$

In all cases, the labeling pattern is designed so that labels assigned to consecutive vertices are consecutive integers, while the vertices connecting the cycles and the path receive labels equal to 1, 2, or consecutive integers. Hence, the coprimality condition follows directly from Lemma 2.3(a) and the facts that 1 is relatively prime to every positive integer and 2 is relatively prime to every odd integer. Therefore, the dumbbell graph $D_{m,n,q}$ admits a prime labeling under all conditions not considered in Theorem 3.16.

The obtained results indicate that the minimum coprime number is strongly influenced by structural properties of graphs, particularly vertex degree, connectivity, and graph density. Graphs with higher vertex degrees impose stronger coprimality constraints on adjacent vertices, which generally increase the minimum coprime number. This behavior is evident in mK_n , where every vertex is adjacent to all other vertices within its component, leading to substantially larger minimum coprime numbers than those of cycle-based constructions. In contrast, graphs composed of cycles exhibit fewer adjacency constraints, allowing more efficient coprime labeling patterns.

These findings are consistent with previous studies on minimum coprime labeling. In particular, Asplund and Fox (2017) observed that complete graphs require considerably larger labels than sparse graph families because of their dense adjacency structure. The exact values established in this paper for mC_n , mK_n , and related comb constructions further confirm the significant role of graph density and local connectivity in determining the minimum coprime number. Moreover, unlike several earlier studies that provided only upper bounds or feasible labeling constructions, the present work establishes matching lower and upper bounds, yielding exact minimum coprime numbers for all graph families considered.

Another notable observation concerns the effect of graph decomposition and graph operations on coprime labeling. Although disjoint unions reduce global connectivity, each component still imposes local coprimality constraints that influence the required label set. This phenomenon is particularly apparent in mC_n , where parity considerations dominate the labeling construction, whereas in mK_n the clique structure necessitates the use of increasingly large prime labels. Similarly, the comb constructions investigated in this study preserve coprime labeling patterns because newly introduced adjacent vertices receive labels that remain relatively prime to their neighbors.

These observations suggest that graph invariants such as maximum degree, average degree, connectivity, clique number, chromatic number, degeneracy, and edge density may provide useful indicators for estimating or bounding the minimum coprime number of a graph. Establishing general relationships between minimum coprime numbers and such graph invariants constitutes an interesting direction for future research.

4. CONCLUSIONS

This paper explores the locating This paper has investigated coprime labeling and the minimum coprime number for several graph constructions, particularly graphs obtained from the disjoint union of cycle graphs and complete graphs. We showed that these graph families admit coprime labeling and determined their minimum coprime numbers under certain conditions. In addition, dumbbell graphs were shown to admit prime and coprime labeling for specific parameters. These results enrich the study of prime and coprime labeling in graph theory. The labeling constructions developed in this work also suggest potential extensions to broader classes of graph operations. In particular, the techniques used to construct coprime labelings and establish minimum coprime numbers may be applicable to other graph families, such as corona products, amalgamation, Cartesian products, and join products. Investigating whether similar arithmetic labeling patterns can be employed in these graph operations constitutes an interesting direction for future research.

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