

A Comparative Study of Confidence Intervals for the Parameter of the XShanker Distribution with Applications

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Abstract

Statistical modeling of lifetime and reliability data has gained significant interest in the last few decades because of its potential use in various disciplines including engineering, biomedical sciences, agriculture, and medicine. For portraying complex data behaviors, especially when the hazard rate demonstrates non-constant patterns, traditional distributions such as the exponential and gamma distributions are frequently insufficient. The XShanker distribution offers a simple mixture distribution that retains a single parameter while providing greater flexibility in modeling positively skewed and heterogeneous data than some conventional lifetime distributions. Despite the increased interest in the XShanker distribution, no thorough comparative evaluation of confidence interval (CI) estimation approaches for its parameter has been conducted. Consequently, this work aimed to recommend four CIs for the XShanker distribution parameter. Using simulation studies and application to two actual datasets, the likelihood, Wald, bootstrap-t, and bias-corrected and accelerated (BCa) bootstrap CIs were assessed. Monte Carlo simulations were used to evaluate their performance, with parameter values ranging from 0.2 to 2.5, sample sizes from 10 to 500, 2000 simulation replications and 2000 bootstrap resamples. The empirical coverage rate (ECR) and average width were used as evaluation criteria. Likelihood and Wald intervals typically maintained coverage rates close to the nominal level of 0.95 across various parameter settings, including small sample sizes. Conversely, the bootstrap-t and BCa bootstrap CIs showed significant undercoverage when with empirical coverage rates decreasing to approximately 0.87-0.92 in some scenarios, although typically yielding narrower intervals. For the ECRs of all four methods were increasingly similar and approached 0.95. The proposed methods were applied to two real datasets on bank customer waiting times and vinyl chloride concentrations in groundwater. The maximum likelihood estimate for the waiting-time data was 0.1945 and the four interval methods produced broadly similar 95% confidence intervals, with the bootstrap-based intervals being slightly narrower. For the vinyl chloride data, the likelihood and Wald CIs were narrower than the bootstrap-t and BCa bootstrap CIs. The results of model comparison based on information criteria and graphical analysis confirmed the XShanker distribution provided a satisfactory fit for the two datasets. These empirical findings were consistent with the simulation results. In conclusion, the likelihood and Wald CIs are recommended when reliable coverage is the primary concern, particularly for small samples, whereas the bootstrap-based CIs may be preferred for larger samples when shorter interval widths are desired.

Keywords

Bootstrap Method, Interval Estimation, Likelihood Method, Mixture Distribution, Wald Method

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1. INTRODUCTION

In the last few decades, statistical modeling of lifetime and reliability data has received a lot of attention. This is likely due to its potential use in various disciplines, such as medicine, engineering, biomedical sciences, science, and environmental studies. Traditional distributions such as the exponential, Lindley and gamma distributions are often inadequate for capturing complex data behaviors, particularly when the hazard rate exhibits non-constant patterns.

Researchers use a variety of techniques to either propose new distributions or modify existing distributions to be more flexible to solve the limitation. Several studies have enhanced existing distributions by introducing additional parameters, thereby increasing their flexibility and broadening their applicability. For example, Alzoubi et al. (2022) proposed the Loai distribution as a two-parameter mixture distribution merging the gamma distribution with parameters and the Lindley distribution with parameter utilizing mixing weights and respectively. The Darna distribution was created by Shraa and Al-Omari

(2019) as a two-parameter mixture of the exponential distribution with parameters and the gamma distribution with parameters with mixing weights identical to those of the Loai distribution, characterized by the parameters and However, the implementation of adding parameters to the probability distribution may lead to more complicated parameter estimation.

An alternative approach to increasing distribution flexibility is to construct one-parameter mixture distributions without adding further parameters. Examples include the Shanker (Shanker, 2015b), Akash (Shanker, 2015a), Aradhana (Shanker, 2016b), Rani (Shanker, 2017c), Gharaibeh (Gharaibeh, 2021), Iwueze (Elechi et al., 2022), Juchez (Echebiri and Mbegbu, 2022), and Ola (Al-Ta'ani and Gharaibeh, 2023) distributions. These distributions aim to preserve inferential simplicity while allowing greater variation in density shape and tail behavior than standard one-parameter distributions. In this context, Etaga et al. (2023a) proposed the XShanker distribution as a mixture of the exponential and Shanker distributions with a common parameter.

The motivation for examining this distribution is not simply to introduce another one-parameter distribution to the existing literature. Instead, the XShanker distribution provides a useful compromise between model flexibility and inferential simplicity. Its mixture structure allows the shape and tail behavior of the density to differ from those of the exponential and Shanker distributions, while retaining only one parameter to be estimated. This characteristic is especially relevant for positively skewed lifetime and environmental data, where an exponential distribution could decline rapidly, and more complex distributions may result in unstable estimations, particularly in small samples. Moreover, the XShanker distribution provides closed-form expressions for its probability density, survival, hazard and moment functions, therefore it is analytically tractable and computationally convenient. The consideration of the XShanker distribution as a reasonable alternative to model heterogeneous waiting time, reliability, and concentration data is practically and theoretically motivated by these properties.

Although the XShanker distribution has attractive structural and computational properties, previous research has focused mainly on its statistical characteristics, point estimation, and goodness-of-fit performance. Point estimates alone do not quantify the uncertainty associated with the unknown parameter, and this limitation is particularly important for small and moderate samples, which are common in reliability, environmental, and biomedical applications. Moreover, because the sampling distribution of the maximum likelihood estimator may be asymmetric in finite samples, confidence intervals (CIs) based solely on the normal approximation may not provide satisfactory coverage.

Recent studies have continued to emphasize the importance of interval estimation for the parameter of lifetime distributions. Likelihood, Wald, bootstrap-t, and bias-corrected and accelerated (BCa) bootstrap CIs have recently been investigated for several one-parameter lifetime models, including the (Srisuradetchai and Panichkitkosolkul, 2024), Zegh-

doudi (Tulyanitikul and Panichkitkosolkul, 2025), RamAwadh (Panichkitkosolkul, 2025), Fav-Jerry (Saengrawee and Panichkitkosolkul, 2025), and XLindley (Thongjub and Panichkitkosolkul, 2026) distributions. These studies generally demonstrate that the performance of an interval estimator depends strongly on sample size and distributional characteristics.

A systematic comparison of likelihood, Wald, and bootstrap-based CIs is therefore required to determine whether the theoretical simplicity of the XShanker distribution also leads to reliable inferential performance. To address this gap, the present study evaluates four CI methods the likelihood, Wald, bootstrap-t, and BCa bootstrap intervals-under a broad range of parameter values and sample sizes. Their performance is assessed in terms of empirical coverage rate (ECR) and average width (AW), and their practical usefulness is illustrated using two real datasets. To the best of our knowledge, this is the first comprehensive investigation of interval estimation for the parameter of the XShanker distribution.

2. EXPERIMENTAL SECTION

The Shanker and XShanker distributions and their statistical properties are examined in this section. Maximum likelihood estimation is used to develop the point estimation of the XShanker distribution parameter. The formation of CIs for the XShanker distribution parameter is detailed at the end of this section.

2.1 The Shanker and XShanker Distributions

The Shanker distribution is a continuous one-parameter model for non-negative random variables, obtained as a mixture of exponential and gamma components. In this formulation, the exponential component has rate parameter θ , whereas the gamma component has shape parameter 2 and rate parameter θ . If X follows the Shanker distribution with parameter θ , its probability density function (pdf) can be expressed through the mixture representation in Equation (1):

$$f_{\text{Shanker}}(x; \theta) = w_1 \cdot f_{\text{Exp}}(x; \theta) + (1 - w_1) \cdot f_{\text{Gam}}(x; \theta, 2), \quad (1)$$

where $w_1 = \theta^2 / (\theta^2 + 1)$. This leads to the closed-form density in Equation (2):

$$f_{\text{Shanker}}(x; \theta) = \frac{\theta^2}{\theta^2 + 1} (\theta + x) e^{-\theta x}, \quad x > 0, \theta > 0 \quad (2)$$

Etaga et al. (2023a) introduced the XShanker distribution, a one-parameter continuous distribution resulting from the merging of exponential and Shanker distributions. The XShanker distribution was created as a more adaptable branch of the Shanker distribution, each weighted by their respective mixing weight $w_2 = 1/(\theta + 1)$, as specified in Equation (3):

$$f_{\text{XShanker}}(x; \theta) = w_2 \cdot f_{\text{Exp}}(x; \theta) + (1 - w_2) \cdot f_{\text{Shanker}}(x; \theta) \quad (3)$$

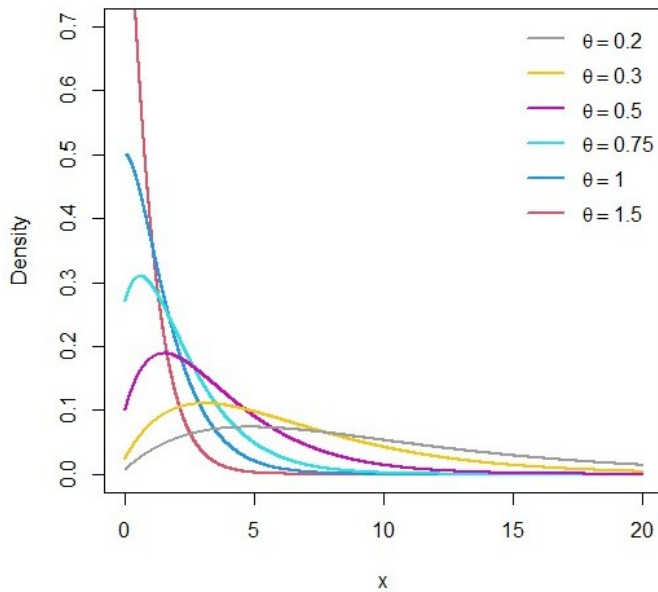


Figure 1. Probability Density Curves of the Shanker Distribution

where $f_{\text{Exp}}(x; \theta)$ is the pdf of an exponential distribution with rate parameter θ and $f_{\text{Shanker}}(x; \theta)$ is shown in Equation (2). Thus, the pdf of the XShanker distribution can be specified in Equation (4):

$$f_{\text{XShanker}}(x; \theta) = \frac{\theta^2}{(\theta^2 + 1)^2} (\theta^3 + 2\theta + x) e^{-\theta x}; \quad x > 0, \theta > 0 \quad (4)$$

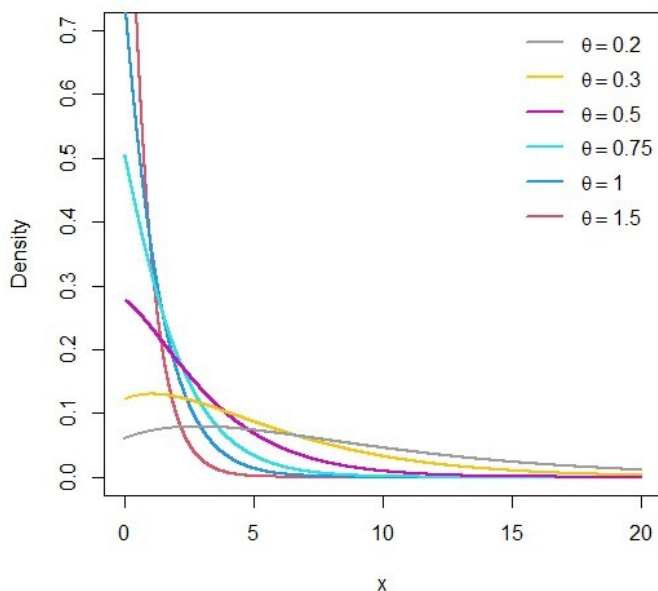


Figure 2. Probability Density Curves of the XShanker Distribution

The probability density curve of the XShanker Distribution is shown in Figure 2. The XShanker distribution offers higher skewness and heavier tails when compared to Shanker distribution, though it depends on the parameter values. When traditional distributions might be inadequate due to variations in the parameter considerably impacting the peak and tail characteristics of the pdf, the XShanker distribution may be more appropriate for modeling complex lifetime data.

Several statistical properties of the XShanker distribution are presented below. These properties describe its location, dispersion, shape, survival behavior, and instantaneous failure risk, thereby providing a theoretical basis for its use in lifetime and reliability applications. The r^{th} raw moment of X following the XShanker distribution is expressed in Equation (5):

$$\mu'_r = E(X^r) = \frac{\theta^2}{(\theta^3 + 1)^2} \left\{ (\theta^3 + 2\theta) \frac{1}{\theta^{r+1}} \Gamma(r+1) + \frac{1}{\theta^{r+2}} \Gamma(r+2) \right\}, \quad r = 1, 2, 3, \dots \quad (5)$$

where $\Gamma(\cdot)$ is the gamma function. The mean (μ) and variance (σ^2) of the XShanker distribution are specified in Equations (6) and (7), respectively:

$$E(X) = \mu = \mu'_1 = \frac{\theta^4 + 2\theta^2 + 2}{\theta(\theta^2 + 1)^2} \quad (6)$$

and

$$Var(X) = \sigma^2 = \mu'_2 - (\mu'_1)^2 = \frac{2\theta^7 + 4\theta^5 + 6\theta^4 + 2\theta^3 + 4\theta^2 - \theta^8}{\theta^2(\theta^2 + 1)^4} \quad (7)$$

The moment generating function is in Equation (8):

$$M_X(t) = \frac{\theta^6 - \theta^5 t + 2\theta^4 - 2\theta^3 t + \theta^2}{\theta^6 - 2\theta^5 t + \theta^4 t^2 + 2\theta^4 - 4\theta^3 t + 2\theta^2 t^2 + \theta^2 - 2\theta t + t^2}, \quad t < \theta \quad (8)$$

The survival and hazard rate functions are given by Equations (9) and (10), respectively, by

$$S(x) = \left[1 + \frac{\theta x}{(\theta^2 + 1)^2} \right] e^{-\theta x} \quad (9)$$

and

$$h(x) = \frac{f_{\text{XShanker}}(x; \theta)}{S(x; \theta)} = \frac{\theta^2(\theta^3 + 2\theta + x)}{\theta^4 + 2\theta^2 + \theta x + 1} \quad (10)$$

These expressions provide useful interpretations of the behavior of the XShanker distribution. The r^{th} moment gives a general framework for deriving measures that describe the location, variability, asymmetry, and tail behavior of the distribution. In particular, the first moment represents the expected lifetime or average waiting time, whereas the variance measures the degree of dispersion around this expected value. The closed-form expressions for the mean and variance allow these characteristics to be evaluated directly for different values of θ .

The moment generating function provides a compact representation of the moments of the distribution and can be used to derive additional distributional measures when required. The survival function, $S(x)$, represents the probability that a lifetime or waiting time exceeds a specified value x . In reliability applications, it describes the probability that a component

remains operational beyond time x , whereas in waiting-time studies, it represents the probability that a customer waits longer than x time units.

The hazard rate function, $h(x)$, describes the instantaneous risk of failure at time x , conditional on survival up to that time. For the XShanker distribution, the hazard rate is increasing in x , approaching θ as x becomes large. This behavior makes the model suitable for systems or individuals whose instantaneous failure risk increases over time and then gradually stabilizes. Taken together, these properties provide a practical basis for using the XShanker distribution to model positively skewed lifetime, waiting-time, and environmental data.

2.2 Estimation of the Distribution Parameter

The maximum likelihood (ML) approach is used to calculate the XShanker distribution parameter θ . Given a random sample $\mathbf{x} = (x_1, x_2, \dots, x_n)$, the likelihood function is formed from the joint density, and the matching log-likelihood is acquired for computational expediency. For a detailed description of the ML estimation procedure, readers are referred to [Sungboonchoo \(2025\)](#). Numerical optimization procedures are utilized to acquire the ML estimate (MLE) as the resulting likelihood equation does not permit a closed-form solution. The MLE in this research is computed utilizing the Newton-Raphson algorithm applied via the `maxLik` package ([Henningesen and Toomet, 2011](#)) in R statistical software ([Team, 2021](#)).

The standard regularity conditions for ML inference hold for the XShanker distribution when the true parameter θ_0 is an interior point of $\Theta = (0, \infty)$. The support $x > 0$ is independent of θ , and the density is positive and continuously differentiable with respect to θ . The distribution is identifiable because different values of θ determine different exponential decay rates. In addition, the score function and its derivatives have finite expectations owing to the exponentially decreasing tail of the distribution, which permits differentiation under the integral sign. The Fisher information is therefore finite and positive for $\theta > 0$ ([Lehmann and Casella, 1998](#)).

Under these conditions, the MLE is consistent and asymptotically normal ([Casella and Berger, 2002](#)): $\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, I^{-1}(\theta_0))$. This result supports the use of the observed information for estimating the standard error and constructing the Wald CI. Nevertheless, because the approximation is asymptotic, its accuracy may decrease for small samples, thereby motivating the inclusion of likelihood and bootstrap alternatives.

2.3 Interval Estimation Methods

This subsection focuses on interval estimation for the parameter of the XShanker distribution. The CIs considered here are developed under both asymptotic likelihood theory and bootstrap resampling frameworks.

2.3.1 Likelihood Confidence Interval

According to the likelihood function, likelihood CI provides an inferential approach for quantifying uncertainty in the estimated parameter. The likelihood function in this study quantifies the plausibility of the observed data under the XShanker distribution and is a function of an unknown parameter. Finding a range of unknown parameter values that augment the likelihood function while sustaining a specified confidence level is the core concept behind the likelihood CI. The estimator is obtained by optimizing the log-likelihood function with respect to the parameter. A likelihood CI is often preferred because it provides reliable coverage performance without depending on normal approximation. Under MLE, the interval is obtained by identifying the range of parameter values that remain sufficiently supported by the likelihood function around the MLE. The likelihood approach relies on the likelihood ratio, denoted by $\lambda(\theta)$ in Equation (11), to construct the CI:

$$\lambda(\theta) = \frac{L(\theta | \mathbf{x})}{L(\hat{\theta} | \mathbf{x})} \quad (11)$$

Wilks' theorem asserts that $-2 \log \lambda(\theta)$ is asymptotically distributed as a chi-square random variable, with degrees of freedom equal to the number of parameters being tested, which is typically one in this study ([Severini, 2000](#)). Consequently, the $(1 - \alpha)100\%$ likelihood interval estimate of θ is expressed by Equation (12):

$$\left\{ \theta \mid -2 \log \frac{L(\theta | \mathbf{x})}{L(\hat{\theta} | \mathbf{x})} \leq \chi_{1, 1-\alpha}^2 \right\} = \left\{ \theta \mid -2 \log \left[\frac{\theta^{2n} (\theta^2 + 1)^{2n}}{\hat{\theta}^{2n} (\hat{\theta}^2 + 1)^{2n}} \cdot \frac{\prod_{i=1}^n (\theta^3 + 2\theta + x_i)}{\prod_{i=1}^n (\hat{\theta}^3 + 2\hat{\theta} + x_i)} \cdot \exp \left(-\theta \sum_{i=1}^n x_i + \hat{\theta} \sum_{i=1}^n x_i \right) \right] \leq \chi_{1, 1-\alpha}^2 \right\} \quad (12)$$

where $\chi_{1, 1-\alpha}^2$ represents the $(1 - \alpha)$ -quantile of the chi-square distribution with one degree of freedom. Figure 3 presents the plot of $-2 \log \lambda(\theta)$ against θ , together with the critical value $\chi_{0.95, 1}^2$ and the corresponding 95% likelihood CI, based on a random sample of size 100 from the XShanker distribution with $\theta = 1$.

The reliance is grounded in Wilks' theorem since the cut-point for creating a likelihood CI often encompasses the utilization of an asymptotic distribution such as the chi-square distribution. Thus, the efficacy of the likelihood CI in estimating the true parameter values hinges on the notion that the sample size is large enough for the asymptotic estimate to be valid.

2.3.2 Wald Confidence Interval

The asymptotic distribution of the estimator, commonly originated from the MLE, is used to compute a Wald CI. It delineates the interval centered at the point estimate and utilizes the estimated standard error and the appropriate quantile from the standard normal distribution to determine its width. Although the Wald CI is widely used in parametric inference

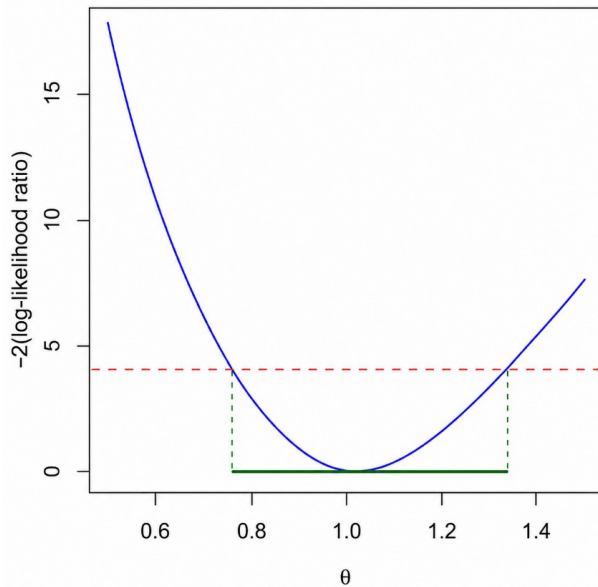


Figure 3. The Plot of $-2\log \lambda(\theta)$ Against θ

because of its simplicity, its accuracy may deteriorate for small samples or when the sampling distribution of the estimator is non-normal. The Wald CI is derived from a second-order Taylor approximation of the log-likelihood function, $L(\theta | \mathbf{x})$, in a neighborhood of the MLE $\hat{\theta}$. Since the first-order term is zero at the MLE, the approximation is given by Equations (13)-(15):

$$\log L(\theta | \mathbf{x}) \approx \log L(\hat{\theta} | \mathbf{x}) + (\theta - \hat{\theta}) \left. \frac{\partial}{\partial \theta} \log L(\theta | \mathbf{x}) \right|_{\theta=\hat{\theta}} + \frac{1}{2}(\theta - \hat{\theta})^2 \left. \frac{\partial^2}{\partial \theta^2} \log L(\theta | \mathbf{x}) \right|_{\theta=\hat{\theta}} \quad (13)$$

$$\log \frac{L(\theta | \mathbf{x})}{L(\hat{\theta} | \mathbf{x})} \approx \frac{1}{2}(\theta - \hat{\theta})^2 \left. \frac{\partial^2}{\partial \theta^2} \log L(\theta | \mathbf{x}) \right|_{\theta=\hat{\theta}} \quad (14)$$

$$-2 \log \frac{L(\theta | \mathbf{x})}{L(\hat{\theta} | \mathbf{x})} \approx (\theta - \hat{\theta})^2 I(\hat{\theta}) \quad (15)$$

where $I(\hat{\theta})$ is the estimated Fisher information. The Wald statistic can be regarded as a large-sample approximation to the likelihood ratio test statistic, leading to a quadratic representation of the log-likelihood ratio under standard regularity conditions (Pawitan, 2001).

For the XShanker distribution, the observed information $J(\hat{\theta})$ in Equation (16) evaluated at the MLE can be expressed as:

$$J(\hat{\theta}) = - \left. \frac{\partial^2}{\partial \theta^2} \log L(\theta | \mathbf{x}) \right|_{\theta=\hat{\theta}} = - \frac{2n}{\hat{\theta}^2} - \frac{4n}{(\hat{\theta}^2 + 1)} + \frac{8n\hat{\theta}^2}{(\hat{\theta}^2 + 1)^2} + \sum_{i=1}^n \frac{6\hat{\theta}(\hat{\theta}^3 + 2\hat{\theta} + x_i) - (3\hat{\theta}^2 + 2)^2}{(\hat{\theta}^3 + 2\hat{\theta} + x_i)^2} \quad (16)$$

and the $(1 - \alpha)100\%$ Wald CI for θ is shown by Equation (17):

$$\left(\hat{\theta} - z_{1-(\alpha/2)} [J(\hat{\theta})]^{-1/2}, \hat{\theta} + z_{1-(\alpha/2)} [J(\hat{\theta})]^{-1/2} \right) \quad (17)$$

where $z_{1-(\alpha/2)}$ denotes the $(1 - (\alpha/2))^{\text{th}}$ quantile of the standard normal distribution.

2.3.3 Bootstrap-t Confidence Interval

The bootstrap-t CI is a resampling-based approach that accounts for variability in the estimator through studentization. This approach creates the interval based on the empirical distribution of a studentized statistic acquired by standardizing bootstrap estimates using their matching standard errors, which is dissimilar to standard asymptotic methods.

Let $\hat{\theta}$ denote the MLE obtained from the observed sample $\mathbf{x} = (x_1, x_2, \dots, x_n)$, and let $\widehat{SE}(\hat{\theta}) = [J(\hat{\theta})]^{-1/2}$ denote its estimated standard error obtained from the observed information. The algorithm for constructing a $(1 - \alpha)100\%$ bootstrap-t CI is as follows.

1. Estimate the XShanker distribution parameter from the observed sample to obtain $\hat{\theta}$ and $\widehat{SE}(\hat{\theta})$.
2. For $b = 1, \dots, B$, generate a parametric bootstrap sample $\mathbf{x}^{*(b)} = (x_1^{*(b)}, x_2^{*(b)}, \dots, x_n^{*(b)})$ of size n from the XShanker distribution with parameter $\hat{\theta}$.
3. For each bootstrap sample, compute the ML estimate $\hat{\theta}^{*(b)}$ and its standard error $\widehat{SE}(\hat{\theta}^{*(b)})$ from the corresponding observed information.
4. Calculate the studentized bootstrap statistic using Equation (18):

$$T^{*(b)} = \frac{\hat{\theta}^{*(b)} - \hat{\theta}}{\widehat{SE}(\hat{\theta}^{*(b)})}, \quad b = 1, 2, \dots, B \quad (18)$$

5. Let $t_{\alpha/2}^*$ and $t_{1-(\alpha/2)}^*$ denote the empirical $\alpha/2$ and $1 - (\alpha/2)$ quantiles, respectively, of the bootstrap statistics $T^{*(1)}, T^{*(2)}, \dots, T^{*(B)}$. The bootstrap-t CI is given by Equation (19):

$$\left(\hat{\theta} - t_{1-(\alpha/2)}^* [J(\hat{\theta})]^{-1/2}, \hat{\theta} - t_{\alpha/2}^* [J(\hat{\theta})]^{-1/2} \right) \quad (19)$$

This approach can yield more accurate CIs, particularly when the sampling distribution is skewed (Efron and Tibshirani, 1993). However, its performance depends on dependable estimation of the standard error for each bootstrap sample and usually requires greater computational effort compared to likelihood and Wald methods. Further details on the bootstrap-t CI can be located in Tulyanitikul and Panichkitkosolkul (2025).

2.3.4 Bias-corrected and Accelerated Bootstrap Confidence Interval

An enhanced resampling-based technique that takes bias and skewness in the estimator's sampling distribution into account

is the bias-corrected and accelerated (BCa) bootstrap CI. In contrast to conventional percentile-based techniques, the BCa bootstrap approach uses an acceleration parameter and a bias-correction factor to modify the interval bounds. This improves the accuracy of the resultant interval, especially in small to intermediate samples and when asymmetry is present.

Let $\hat{\theta}^{*(1)}, \hat{\theta}^{*(2)}, \dots, \hat{\theta}^{*(B)}$ be the ML estimates obtained from the same B parametric bootstrap samples. The bias-correction constant is computed using Equation (20):

$$\hat{z}_0 = \Phi^{-1} \left[\frac{1}{B} \sum_{b=1}^B I(\hat{\theta}^{*(b)} < \hat{\theta}) \right] \quad (20)$$

where $\Phi^{-1}(\cdot)$ is the inverse cumulative distribution function of the standard normal distribution and $I(\cdot)$ denotes the indicator function. The acceleration constant is estimated using the jackknife method. Let $\hat{\theta}_{(-i)}$ denote the ML estimate computed after deleting the i^{th} observation from the original sample, for $i = 1, 2, \dots, n$, and define the average jackknife estimate using Equation (21):

$$\hat{\theta}_{(\cdot)} = \frac{1}{n} \sum_{i=1}^n \hat{\theta}_{(-i)} \quad (21)$$

The acceleration constant is defined by Equation (22):

$$\hat{a} = \frac{\sum_{i=1}^n (\hat{\theta}_{(\cdot)} - \hat{\theta}_{(-i)})^3}{6 \left[\sum_{i=1}^n (\hat{\theta}_{(\cdot)} - \hat{\theta}_{(-i)})^2 \right]^{3/2}} \quad (22)$$

The adjusted percentile levels are given by Equations (23) and (24):

$$\alpha_L = \Phi \left[\hat{z}_0 + \frac{\hat{z}_0 + z_{\alpha/2}}{1 - \hat{a}(\hat{z}_0 + z_{\alpha/2})} \right] \quad (23)$$

and

$$\alpha_U = \Phi \left[\hat{z}_0 + \frac{\hat{z}_0 + z_{1-(\alpha/2)}}{1 - \hat{a}(\hat{z}_0 + z_{1-(\alpha/2)})} \right] \quad (24)$$

The $(1 - \alpha)100\%$ BCa bootstrap CI for θ is given by Equation (25):

$$(\hat{\theta}_{(\alpha_L)}^*, \hat{\theta}_{(\alpha_U)}^*) \quad (25)$$

where $\hat{\theta}_{(q)}^*$ denotes the empirical q^{th} quantile of the bootstrap estimates. The bias-correction term accounts for systematic displacement of the bootstrap distribution relative to $\hat{\theta}$, whereas the acceleration term adjusts for changes in the standard error as the data vary. A more detailed description of the BCa bootstrap CI is provided in Lemonte (2025) and Tulyanitikul and Panichkitkosolkul (2025).

2.3.5 Computational Implementation

All numerical computations were performed using R statistical software. For each bootstrap sample, the same estimation procedure, parameter constraints, starting-value rule, and convergence criterion as those used for the original sample were applied. Across all Monte Carlo scenarios, the convergence status of each numerical optimization was examined using the default convergence criteria of the R optimization routine. Bootstrap fits that failed to converge or produced non-positive observed-information values were regenerated. Only converged estimates satisfying the parameter constraint $\theta > 0$ and yielding finite positive standard errors were retained for the analysis.

For each Monte Carlo sample, $B = 2000$ parametric bootstrap replications were generated for the bootstrap-t and BCa bootstrap methods. Any bootstrap replicate for which the numerical optimization failed to converge or produced an invalid observed-information value was regenerated until B valid bootstrap estimates were obtained.

3. RESULTS AND DISCUSSIONS

3.1 Simulation Results

To assess the execution of the recommended CIs for the XShanker distribution parameter, a Monte Carlo simulation study was carried out in the R statistical software. In this research, the empirical coverage rate (ECR) and average width (AW) of 95% CIs were investigated because they directly represent the two principal objectives of CI estimation. The ECR measures the reliability of an interval in attaining the nominal confidence level, whereas the AW reflects its precision. These criteria should be interpreted jointly, since a shorter interval is desirable only when it maintains adequate coverage.

For a broad range of parameter values and sample sizes, random samples were produced from the XShanker distribution. The parameter values were selected to represent distinct distributional shapes. Small values of θ generate more dispersed and strongly right-skewed observations with longer upper tails, whereas larger values produce distributions that are more concentrated near zero. The values $\theta = 0.2$ and $\theta = 0.75$ also approximate the estimates obtained from the two empirical applications. The sample sizes $n = 10, 20$, and 30 represent small-sample situations; $n = 50$ and 100 represent moderate samples; and $n = 200$ and 500 were included to examine convergence toward asymptotic behavior. The quantity of simulation replicates was set at 2,000 for every combination of (θ, n) . The study used $B = 2,000$ bootstrap replications for the bootstrap-based CIs.

Tables 1 and 2 exhibit the simulation findings for the XShanker distribution parameter's 95% CIs, and Figures 4 and 5 show the ECRs and AWs graphically. The findings demonstrate that, for all actual parameter values, the ECRs of all CI techniques tend to approach the desired confidence coefficient of 0.95 as sample size grows. For small sample sizes, the coverage performance of the Wald and likelihood CIs are relatively constant across all parameter configurations. For a number of scenarios,

Table 1. Simulation Results for 95% Interval Estimation of the Shanker Distribution Parameter ($\theta = 0.2, 0.3, 0.5,$ and 0.75)

θ	n	Empirical Coverage Rate (ECR)				Average Width (AW)			
		Likelihood	Wald	Boot-t	BCa	Likelihood	Wald	Boot-t	BCa
0.2	10	0.9455	0.9470	0.8815	0.8745	0.1767	0.1763	0.1558	0.1625
	20	0.9580	0.9600	0.9240	0.9255	0.1223	0.1221	0.1149	0.1165
	30	0.9505	0.9525	0.9315	0.9255	0.0988	0.0988	0.0944	0.0952
	50	0.9450	0.9430	0.9295	0.9315	0.0761	0.0761	0.0742	0.0745
	100	0.9415	0.9400	0.9335	0.9340	0.0538	0.0538	0.0532	0.0533
	200	0.9460	0.9455	0.9395	0.9395	0.0379	0.0379	0.0374	0.0375
	500	0.9480	0.9460	0.9450	0.9425	0.0240	0.0240	0.0239	0.0239
0.3	10	0.9465	0.9485	0.8940	0.8920	0.2566	0.2557	0.2273	0.2382
	20	0.9505	0.9515	0.9190	0.9185	0.1770	0.1768	0.1658	0.1683
	30	0.9525	0.9460	0.9270	0.9310	0.1429	0.1427	0.1368	0.1378
	50	0.9545	0.9520	0.9335	0.9400	0.1102	0.1102	0.1070	0.1075
	100	0.9460	0.9465	0.9400	0.9385	0.0782	0.0781	0.0770	0.0773
	200	0.9495	0.9530	0.9450	0.9410	0.0549	0.0549	0.0544	0.0546
	500	0.9360	0.9385	0.9395	0.9405	0.0348	0.0348	0.0345	0.0347
0.5	10	0.9605	0.9605	0.8940	0.9025	0.4127	0.4084	0.3596	0.3912
	20	0.9470	0.9465	0.9125	0.9190	0.2847	0.2834	0.2668	0.2746
	30	0.9535	0.9595	0.9375	0.9355	0.2303	0.2297	0.2190	0.2237
	50	0.9580	0.9580	0.9385	0.9430	0.1767	0.1764	0.1716	0.1734
	100	0.9580	0.9590	0.9485	0.9485	0.1250	0.1249	0.1225	0.1233
	200	0.9495	0.9485	0.9420	0.9455	0.0877	0.0876	0.0863	0.0867
	500	0.9520	0.9525	0.9440	0.9485	0.0554	0.0554	0.0550	0.0552
0.75	10	0.9475	0.9605	0.8970	0.9060	0.6599	0.6451	0.5623	0.6433
	20	0.9460	0.9535	0.9160	0.9215	0.4413	0.4365	0.4050	0.4267
	30	0.9585	0.9570	0.9395	0.9405	0.3517	0.3492	0.3301	0.3421
	50	0.9405	0.9450	0.9250	0.9255	0.2722	0.2710	0.2641	0.2694
	100	0.9445	0.9465	0.9405	0.9385	0.1898	0.1894	0.1862	0.1883
	200	0.9515	0.9505	0.9430	0.9450	0.1337	0.1335	0.1327	0.1337
	500	0.9515	0.9540	0.9525	0.9510	0.0843	0.0842	0.0837	0.0842

these two approaches frequently slightly exceed the nominal confidence level.

Conversely, the bootstrap-t and BCa bootstrap CIs offered considerable under-coverage when the sample size and parameter are small. For $n = 10$ and $n = 20$, this situation is more evident, where the bootstrap-based CIs are typically narrower, thereby reducing the ECRs. Coverage performance of both bootstrap approaches improves significantly as the sample size grows. For $n \geq 50$, their ECRs are generally consistent with those of the likelihood and Wald CIs across the full range of parameter values. In terms of AW, the bootstrap-t CI consistently produces the smallest values among the methods considered, with the BCa bootstrap CI ranking second. In contrast, the likelihood and Wald CIs tend to produce broader intervals, especially as θ increases.

The relatively narrow BCa bootstrap CI arises from their use of adjusted empirical quantiles rather than symmetric normal-theory limits. The bias-correction factor accounts for systematic displacement of the bootstrap distribution, whereas

the acceleration constant adjusts for skewness and changes in the estimator’s variability (Efron and Tibshirani, 1993; Davison and Hinkley, 1997). These corrections allow the lower and upper limits to adapt separately to the shape of the bootstrap distribution and may produce shorter intervals than the likelihood and Wald methods. As the sample size increases, the bootstrap distribution becomes more stable and less skewed, enabling the BCa bootstrap method to retain relatively short widths while attaining coverage rates close to the nominal level. For very small samples, however, the correction terms may be estimated imprecisely, which helps explain the observed undercoverage. As expected, larger sample sizes lead to smaller AWs across all CI methods, suggesting improved precision of the parameter estimates.

Figure 5 provides an illustration of this pattern. The AWs of the Wald CI and the likelihood CI are nearly the same for all sample sizes and parameter settings. This is consistent with the asymptotic relationship between quadratic-approximation-based and likelihood inference. The execution of CI methods

Table 2. Simulation Results for 95% Interval Estimation of the Shanker Distribution Parameter ($\theta = 1, 1.5, 2,$ and 2.5)

θ	n	Empirical Coverage Rate (ECR)				Average Width (AW)			
		Likelihood	Wald	Boot-t	BCa	Likelihood	Wald	Boot-t	BCa
1	10	0.9465	0.9585	0.8940	0.8875	0.9730	0.9455	0.8146	0.9485
	20	0.9515	0.9560	0.9255	0.9290	0.6431	0.6333	0.5850	0.6246
	30	0.9470	0.9540	0.9220	0.9270	0.5117	0.5063	0.4765	0.4969
	50	0.9485	0.9510	0.9320	0.9370	0.3875	0.3851	0.3705	0.3800
	100	0.9430	0.9470	0.9345	0.9340	0.2717	0.2709	0.2657	0.2697
	200	0.9605	0.9605	0.9550	0.9575	0.1913	0.1910	0.1882	0.1900
	500	0.9475	0.9495	0.9415	0.9450	0.1205	0.1204	0.1190	0.1198
1.5	10	0.9475	0.9560	0.8940	0.8715	1.7665	1.7293	1.5003	1.7068
	20	0.9465	0.9580	0.9220	0.9135	1.1546	1.1404	1.0541	1.1071
	30	0.9625	0.9590	0.9380	0.9310	0.9064	0.8982	0.8557	0.8833
	50	0.9495	0.9510	0.9310	0.9340	0.6866	0.6826	0.6618	0.6748
	100	0.9515	0.9525	0.9390	0.9425	0.4778	0.4764	0.4661	0.4716
	200	0.9615	0.9625	0.9580	0.9585	0.3360	0.3355	0.3304	0.3326
	500	0.9480	0.9475	0.9395	0.9420	0.2114	0.2112	0.2098	0.2110
2	10	0.9460	0.9505	0.9015	0.8905	2.4949	2.4627	2.1586	2.4177
	20	0.9465	0.9460	0.9115	0.9055	1.6690	1.6586	1.5376	1.5866
	30	0.9450	0.9510	0.9215	0.9190	1.3383	1.3326	1.2615	1.2852
	50	0.9440	0.9465	0.9305	0.9305	1.0067	1.0039	0.9733	0.9835
	100	0.9610	0.9610	0.9500	0.9455	0.7054	0.7044	0.6902	0.6938
	200	0.9620	0.9620	0.9545	0.9555	0.4959	0.4956	0.4917	0.4943
	500	0.9550	0.9540	0.9515	0.9530	0.3119	0.3118	0.3096	0.3113
2.5	10	0.9395	0.9480	0.8875	0.8650	3.3165	3.2875	2.8590	3.1713
	20	0.9510	0.9545	0.9305	0.9265	2.1745	2.1672	2.0051	2.0638
	30	0.9385	0.9390	0.9155	0.9110	1.7467	1.7430	1.6658	1.6833
	50	0.9470	0.9550	0.9370	0.9350	1.3423	1.3407	1.2951	1.3048
	100	0.9525	0.9520	0.9460	0.9455	0.9327	0.9323	0.9198	0.9230
	200	0.9555	0.9570	0.9450	0.9480	0.6571	0.6569	0.6460	0.6483
	500	0.9510	0.9490	0.9415	0.9425	0.4136	0.4136	0.4092	0.4113

is highly reliant on the sample size as well as underlying distributional characteristics, as shown by the simulation results. Particularly, the likelihood approach performs consistently in small-sample settings, while resampling-based procedures tend to require larger sample sizes to attain coverage close to the nominal level. Furthermore, there is an obvious compromise between interval width and coverage accuracy: the likelihood technique typically produces more consistent coverage at the expense of somewhat wider intervals, whereas bootstrap methods typically provide tighter intervals but it coverages below the nominal level in small-sample settings. Depending on sample size and the intended balance between precision and accuracy, these results offer useful suggestions for selecting appropriate CI techniques.

3.2 Real Data Applications

Using two actual datasets, this section demonstrates how the suggested 95% CI approaches for the XShanker distribution parameter are applied. To demonstrate their performance with

actual data, the likelihood, Wald, bootstrap-t, and BCa bootstrap CIs are analyzed. Comparative analysis was achieved including twenty contending one-parameter distributions as well as the XShanker distribution. All the pdfs are expressed for $x > 0$ and are depicted by a parameter $\theta > 0$. All competing distributions were fitted by MLE under their respective positivity constraints using the same numerical optimization framework and convergence criterion to ensure a consistent comparison.

1) Adya distribution (Shanker and Shukla, 2021)

$$f(x; \theta) = \frac{\theta^3}{\theta^4 + 2\theta^2 + 2}(\theta + x)^2e^{-\theta x}$$

2) Akash distribution (Shanker, 2015b)

$$f(x; \theta) = \frac{\theta^3}{\theta^2 + 2}(1 + x^2)e^{-\theta x}$$

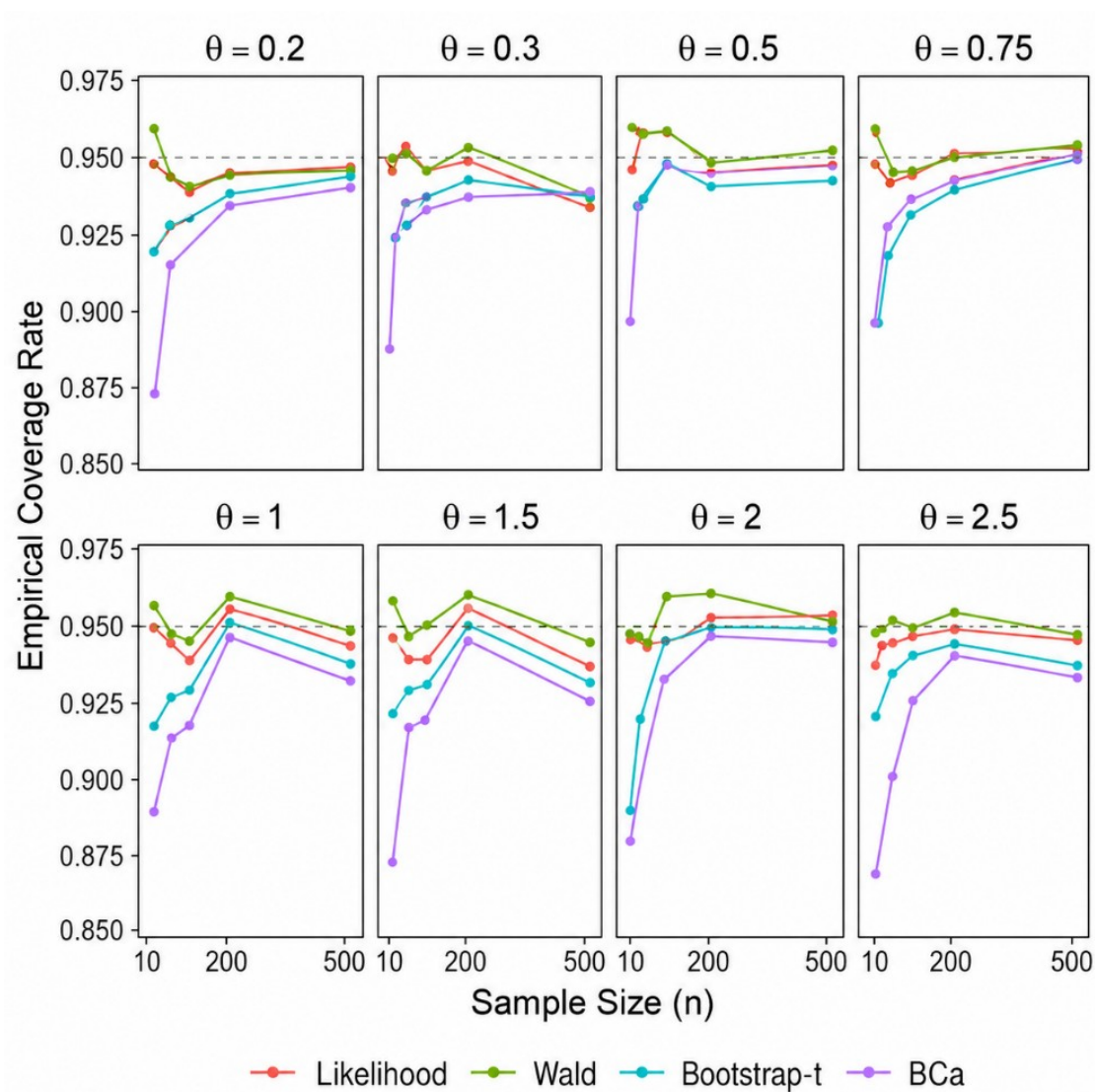


Figure 4. Plots Showing the ECRs of the CIs for the XShanker Distribution Parameter

Table 3. Descriptive Statistics for the Bank Customer Wait Time Data

Sample Sizes	Minimize	Q1	Median	Mean	Q3	Maximum	St.Dev	Skewness	Kurtosis
100	0.800	4.675	8.100	9.877	13.025	38.500	7.237	1.472	5.005

- 3) Akshaya distribution (Shanker, 2017b)
$$f(x;\theta) = \frac{\theta^4}{\theta^3 + 3\theta^2 + 6\theta + 6} (1+x)^3 e^{-\theta x}$$
- 4) Amarendra distribution (Shanker, 2016b)
$$f(x;\theta) = \frac{\theta^4}{\theta^3 + \theta^2 + 2\theta + 6} (1+x+x^2+x^3) e^{-\theta x}$$
- 5) Aradhana distribution (Shanker, 2016a)
$$f(x;\theta) = \frac{\theta^3}{\theta^2 + 2\theta + 2} (1+x)^2 e^{-\theta x}$$
- 6) Arz distribution (Gharaibeh and Al-Ayasrah, 2024)
$$f(x;\theta) = \frac{\theta^8}{\theta^7 + 2\theta^5 + 5040} (x^7 + x^2 + 1) e^{-\theta x}$$
- 7) Chris-Jerry distribution (Onyekwere and Obulezi, 2022)
$$f(x;\theta) = \frac{\theta^2}{\theta + 2} (1 + \theta x^2) e^{-\theta x}$$
- 8) Devya distribution (Shanker, 2016c)
$$f(x;\theta) = \frac{\theta^5}{\theta^4 + \theta^3 + 2\theta^2 + 6\theta + 24} (1+x+x^2+x^3+x^4) e^{-\theta x}$$

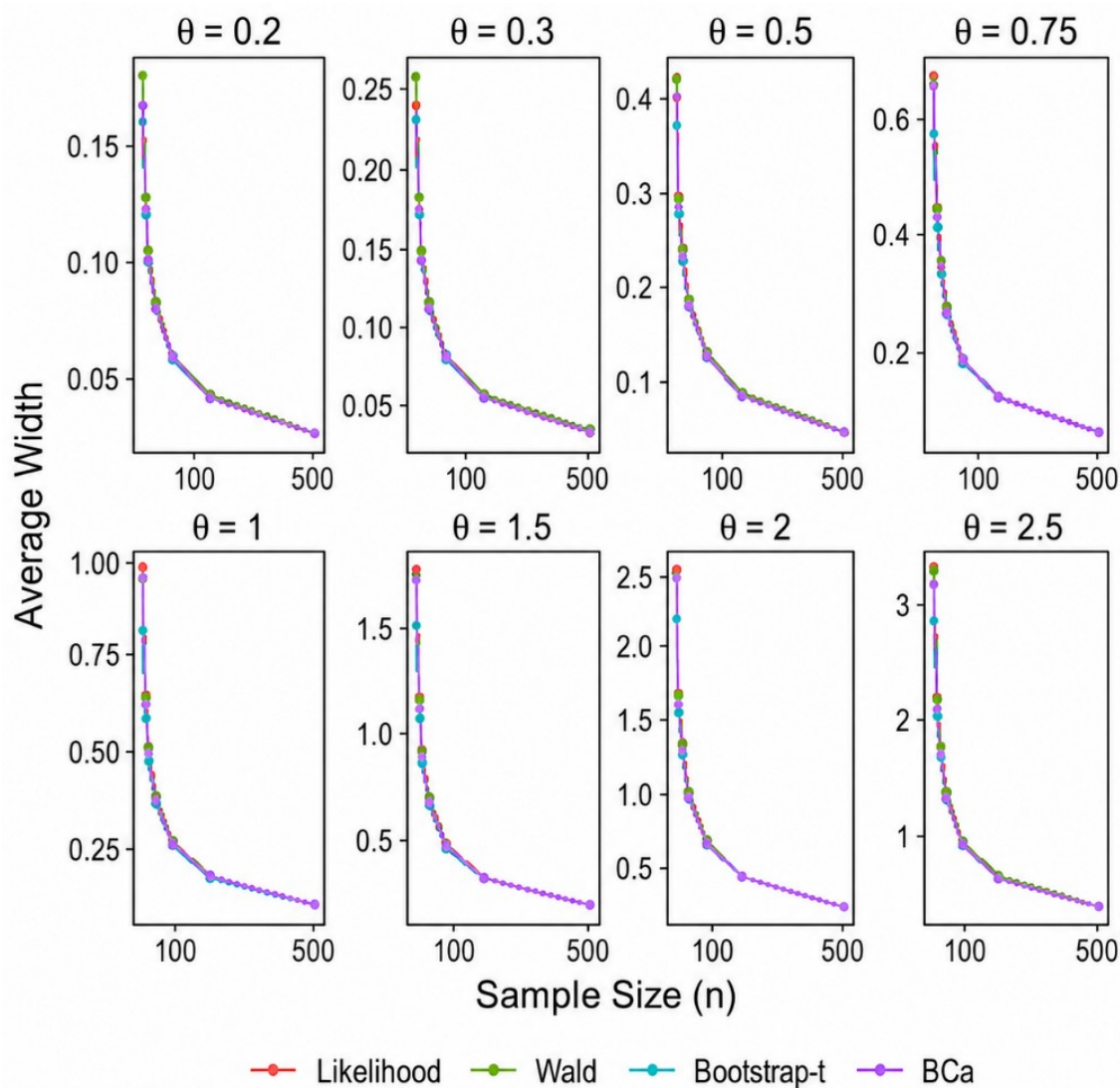


Figure 5. Plots Showing the AWs of the CIs for the XShanker Distribution Parameter

9) Garima distribution (Shanker, 2016d)

$$f(x; \theta) = \frac{\theta}{\theta + 2} (1 + \theta + \theta x) e^{-\theta x}$$

10) Ishita distribution (Shanker, 2017a)

$$f(x; \theta) = \frac{\theta^3}{\theta^3 + 2} (\theta + x^2) e^{-\theta x}$$

11) Iwok-Nwike distribution (Iwok and Nwike, 2021)

$$f(x; \theta) = \frac{\theta^3}{\theta + 2} (x^2 + x) e^{-\theta x}$$

12) Iwueze distribution (Elechi et al., 2022)

$$f(x; \theta) = \frac{\theta^5}{\theta^4 + 2\theta^3 + 6\theta^2 + 12\theta + 24} (1 + x + x^2)^2 e^{-\theta x}$$

13) Lindley distribution (Ghitany et al., 2008)

$$f(x; \theta) = \frac{\theta^2}{\theta + 1} (1 + x) e^{-\theta x}$$

14) Nwike distribution (Nwike et al., 2021)

$$f(x; \theta) = \frac{\theta^3}{12(\theta^2 + 10)} (\theta^3 x^5 + 12) e^{-\theta x}$$

15) Ola distribution (Al-Ta'ani and Gharaibeh, 2023)

$$f(x; \theta) = \frac{\theta^8}{\theta^7 + 6\theta^4 + 5040} (x^7 + x^3 + 1) e^{-\theta x}$$

16) Prakaamy distribution (Shukla, 2018)

$$f(x; \theta) = \frac{\theta^6}{\theta^5 + 120} (1 + x^5) e^{-\theta x}$$

Table 4. Evaluation of Model Fit for Various Distributions Used for the Waiting Time of Bank Customer Data, with Distributions Ranked in Ascending Order Based on Their AIC Values

Distributions	Estimate (SE)	LogLik	AIC	BIC	Rank(AIC)
Shanker	0.1945 (0.0133)	-317.9817	637.9634	640.5686	1
Aradhana	0.2766 (0.0160)	-319.1715	640.3430	642.9482	2
Lindley	0.1983 (0.0138)	-319.4130	640.8260	643.4311	3
Sujatha	0.2846 (0.0163)	-319.8194	641.6387	644.2439	4
Pratibha	0.2892 (0.0166)	-320.1665	642.3329	644.9381	5
Chris-Jerry	0.2791 (0.0167)	-320.3431	642.6862	645.2914	6
Iwok-Nwikpe	0.2909 (0.0168)	-320.4032	642.8065	645.4117	7
Adya	0.2961 (0.0166)	-320.9369	643.8737	646.4789	8
Akash	0.2953 (0.0168)	-320.9646	643.9292	646.5344	9
Ishita	0.3016 (0.0172)	-321.8498	645.6996	648.3047	10
Garima	0.1515 (0.0131)	-324.8188	651.6375	654.2427	11
Akshaya	0.3679 (0.0184)	-324.8588	651.7175	654.3227	12
Amarendra	0.3869 (0.0192)	-327.9529	657.9057	660.5109	13
Rama	0.4017 (0.0199)	-331.6121	665.2242	667.8294	14
Iwueze	0.4762 (0.0212)	-337.0300	676.0600	678.6651	15
Devya	0.4898 (0.0218)	-340.7936	683.5872	686.1924	16
Suja	0.5051 (0.0225)	-346.6270	695.2540	697.8592	17
Nwikpe	0.5740 (0.0244)	-348.3459	698.6919	701.2970	18
Prakaamy	0.6071 (0.0248)	-363.9703	729.9406	732.5458	19
Ola	0.8097 (0.0469)	-401.0715	804.1430	806.7482	20
Arz	0.8099 (0.0286)	-401.2366	804.4733	807.0785	21

The best-fitting model consistent with the least AIC and BIC criteria is specified by bold values

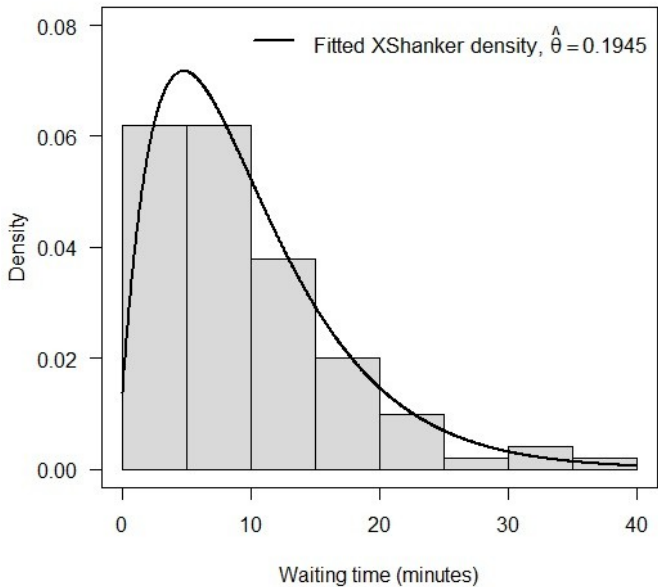


Figure 6. Histogram of the Bank Customer Waiting-Time Data with the Fitted Pdf of the XShanker Distribution

17) Pratibha distribution (Shanker, 2023)

$$f(x;\theta) = \frac{\theta^3}{\theta^3 + \theta + 2}(\theta + x + x^2)e^{-\theta x}$$

18) Rama distribution (Shanker, 2017c)

$$f(x;\theta) = \frac{\theta^4}{\theta^3 + 6}(1 + x^3)e^{-\theta x}$$

19) Suja distribution (Shanker, 2017d)

$$f(x;\theta) = \frac{\theta^5}{\theta^4 + 24}(1 + x^4)e^{-\theta x}$$

20) Sujatha distribution (Shanker, 2016e)

$$f(x;\theta) = \frac{\theta^3}{\theta^2 + \theta + 2}(1 + x + x^2)e^{-\theta x}$$

3.2.1 Waiting Time of Bank Customer Data
Ghitany et al. (2008) examined and analyzed the amount of time (in minutes) 100 bank customers needed to wait for service. The dataset comprises the following data:
0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11, 11, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19, 19.9, 20.6, 21.3, 21.4, 21.9, 23, 27, 31.6, 33.1, 38.5.

Table 5. The 95% and 99% CIs and Their Matching Widths for the Shanker Distribution Parameter Derived From the Bank Customer Data for Wait Time

CI Method	95% CI		99% CI	
	Interval	Width	Interval	Width
Likelihood	(0.1695, 0.2217)	0.0522	(0.1621, 0.2307)	0.0686
Wald	(0.1684, 0.2206)	0.0522	(0.1602, 0.2288)	0.0686
Bootstrap-t	(0.1693, 0.2213)	0.0520	(0.1637, 0.2305)	0.0668
BCa	(0.1689, 0.2195)	0.0506	(0.1640, 0.2289)	0.0649

Table 6. Predictive Probabilities and 95% Parametric Bootstrap CIs for Selected Bank Customer Waiting-Time Events

Waiting-Time Event	Empirical Probability	Estimated Shanker Probability	95% Bootstrap CI	Absolute Error
$X \leq 5$ minutes	0.3100	0.2805	(0.2299, 0.3426)	0.0295
$X \leq 10$ minutes	0.6200	0.5989	(0.5247, 0.6775)	0.0211
$X \leq 15$ minutes	0.8100	0.7995	(0.7345, 0.8596)	0.0105
$X > 20$ minutes	0.0900	0.0943	(0.0573, 0.1401)	0.0043

Note: The 95% bootstrap CIs were obtained using 2,000 parametric bootstrap replications. Absolute error is the absolute difference between the empirical probability and the probability estimated from the fitted Shanker distribution

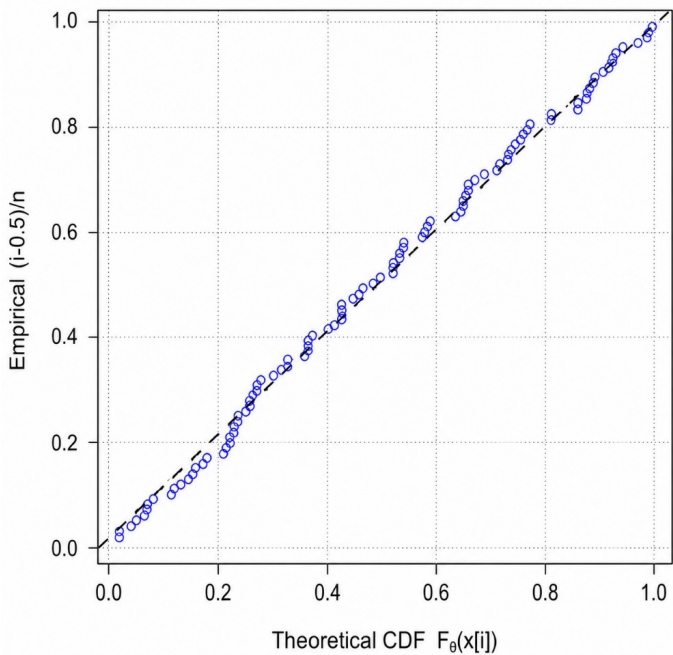


Figure 7. P-P Plot Showing the Data for Bank Customer Wait Time

The descriptive statistics representing the bank customer wait time data are exhibited in Table 3, which is based on a sample size of 100 observations. The data shows a wide dispersion, with waiting times ranging from a minimum of 0.800 to a maximum of 38.500. The first quartile (Q1), median, third quartile (Q3) are 4.675, 8.100, and 13.025, respectively. The

median is lower than the mean waiting time of 9.877, indicating a right-skewed distribution. This is further supported by the positive skewness value of 1.472. Furthermore, the kurtosis value of 5.005 points to a leptokurtic distribution, which implies that the data are more peaked and have heavier tails than a normal distribution. In general, the waiting time data demonstrate a high degree of variability, positive skewness, and heavy-tailed behavior.

The fitted models were compared in terms of the log-likelihood, Akaike information criterion (AIC), and Bayesian information criterion (BIC) to assess their relative adequacy. A distribution with lower values of the AIC and BIC statistics is considered to be preferable. As shown in Table 4, the parameter estimates, their standard errors (SEs), and corresponding model evaluation criteria are reported for the dataset under consideration. Empirical evidence from this dataset further supports the suitability of the XShanker distribution, as it provides lower AIC and BIC values compared to other mixture distributions.

Figure 6 presents the histogram of the bank customer waiting-time data together with the fitted pdf of the XShanker distribution, evaluated at the ML estimate $\hat{\theta} = 0.1945$. The fitted density reflects the concentration of observations at relatively short waiting times and the gradual decline toward the upper end of the data range. It also captures the pronounced positive skewness and the extended right tail of the observed data. This graphical agreement is consistent with the model-comparison results reported in Table 4 and provides additional evidence supporting the suitability of the XShanker distribution for the waiting-time data.

The Kolmogorov–Smirnov (K-S) goodness-of-fit statistic (Corder and Foreman, 2014) was used to compare the em-

Table 7. Descriptive Statistics for the Vinyl Chloride Concentration in Upgradient Groundwater

Sample Sizes	Minimize	Q1	Median	Mean	Q3	Maximum	St.Dev	Skewness	Kurtosis
34	0.100	0.500	1.150	1.879	2.475	8.000	1.952	1.604	5.005

Table 8. Evaluation of Model Fit for Various Distributions Used for the Vinyl Chloride Concentration in Upgradient Groundwater, With Distributions Ranked in Ascending Order Based on Their AIC Values

Distributions	Estimate (SE)	LogLik	AIC	BIC	Rank(AIC)
Shanker	0.7567 (0.0828)	-20.3749	42.7498	44.2761	1
Garima	0.7234 (0.1074)	-20.4066	42.8131	44.3395	2
Ishita	1.1570 (0.0962)	-20.7628	43.5256	45.0520	3
Lindley	0.8534 (0.0948)	-21.0168	44.0336	45.5599	4
Suja	1.9056 (0.1214)	-21.1344	44.2687	45.7951	5
Rama	1.5310 (0.1176)	-21.2364	44.4728	45.9992	6
Prakaamy	2.2843 (0.1244)	-21.2912	44.5823	46.1087	7
Akash	1.1657 (0.1126)	-21.5477	45.0954	46.6217	8
Arz	3.0117 (0.1294)	-21.7411	45.4822	47.0086	9
Ola	3.0240 (0.0469)	-22.0701	46.1402	47.6666	10
Pratibha	1.1452 (0.1025)	-22.2867	46.5733	48.0997	11
Chris-Jerry	1.1644 (0.1321)	-22.6803	47.3606	48.8869	12
Sujatha	1.1461 (0.1160)	-23.1339	48.2679	49.7942	13
Adya	1.1298 (0.0961)	-23.1604	48.3208	49.8472	14
Nwikpe	2.1089 (0.1754)	-23.7821	49.5643	51.0907	15
Amarendra	1.4915 (0.1242)	-23.8456	49.6912	51.2176	16
Devy	1.8538 (0.1307)	-24.1432	50.2864	51.8127	17
Aradhana	1.1329 (0.1182)	-24.3418	50.6836	52.2100	18
Iwueze	1.8142 (0.1364)	-26.9511	55.9022	57.4285	19
Akshaya	1.4534 (0.1300)	-27.0377	56.0753	57.6017	20
Iwok-Nwikpe	1.3791 (0.1405)	-39.2804	80.5609	82.0872	21

The best-fitting model consistent with the least AIC and BIC criteria is specified by bold values

Table 9. The 95% and 99% CIs and Their Matching Widths for the Shanker Distribution Parameter Derived From the Vinyl Chloride Concentration in Upgradient Groundwater

CI Method	95% CI		99% CI	
	Interval	Width	Interval	Width
Likelihood	(0.6075, 0.9341)	0.3266	(0.5655, 0.9966)	0.4311
Wald	(0.5944, 0.9191)	0.3247	(0.5434, 0.9701)	0.4267
Bootstrap-t	(0.5918, 0.9506)	0.3588	(0.5603, 1.0119)	0.4516
BCa	(0.5913, 0.9522)	0.3609	(0.5603, 1.0290)	0.4687

pirical distribution of the bank customer waiting times with the fitted XShanker distribution. The K-S statistic was 0.0489, with a p-value of 0.9703, suggesting that the XShanker distribution provides an adequate fit to the bank customer waiting time data.

Consistent with the K-S test results, the P–P plot in Figure 7 shows that the plotted points closely align with the diagonal reference line, providing additional evidence that the XShanker distribution fits the bank customer waiting time data well. To-

gether with the K-S test and P-P plot results, this indicates that the XShanker distribution provides a satisfactory fit to the data.

For the XShanker parameter, the ML estimate was $\hat{\theta} = 0.1945$. As shown in Table 5, the four 95% CIs were very similar, with widths ranging from 0.0506 to 0.0522. This pattern is consistent with the simulation setting closest to the empirical application, namely $\theta = 0.2$ and $n = 100$, for which the AWs of the likelihood, Wald, bootstrap-t, and BCa bootstrap CIs were 0.0538, 0.0538, 0.0532, and 0.0533, respectively. In both

the simulation and real-data results, the likelihood and Wald intervals had nearly identical widths, while the bootstrap-based intervals were only slightly narrower. Thus, the waiting-time application supports the simulation finding that differences among the four methods become relatively small for moderate to large samples.

To assess the practical performance of the fitted XShanker distribution, probabilities at selected waiting-time thresholds were estimated and compared with the corresponding empirical proportions. Table 6 reports the model-based probabilities, 95% parametric bootstrap CIs, and absolute errors. The estimated probabilities were close to the empirical proportions at all thresholds, with absolute errors ranging from 0.0043 to 0.0295. The closest agreement was observed for waiting times exceeding 20 minutes. Moreover, each empirical probability was contained within the corresponding 95% bootstrap CI. These findings indicate that the fitted XShanker distribution provides probability estimates that are consistent with the observed data and are practically interpretable for the bank customer waiting-time data.

3.2.2 Vinyl Chloride Concentration in Upgradient Groundwater

The second dataset consists of vinyl chloride concentrations (g/L) from clean upgradient groundwater observing wells, as studied by Tolba et al. (2023). The data set includes the following observations:

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4, 0.2.

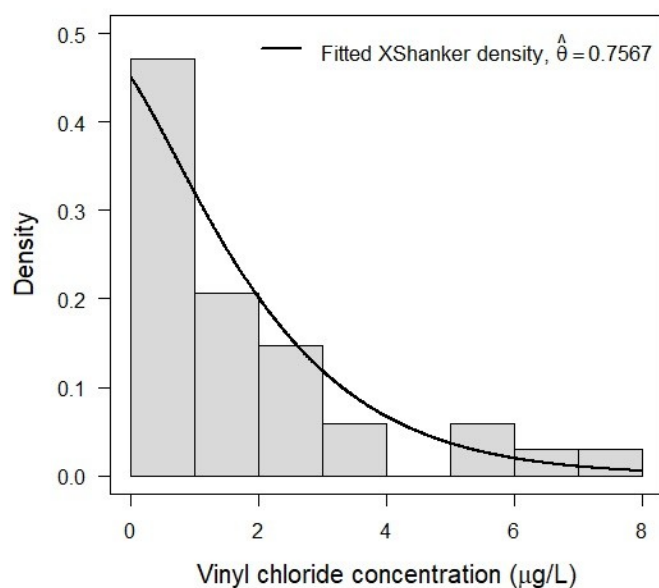


Figure 8. Histogram of the Vinyl Chloride Concentration Data in Upgradient Groundwater with the Fitted Pdf of the XShanker Distribution

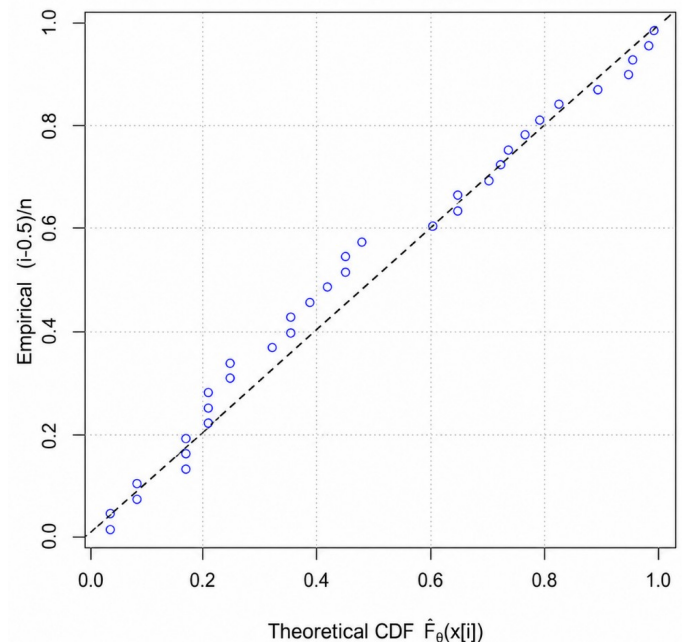


Figure 9. P-P Plot for Vinyl Chloride Concentrations in Upgradient Groundwater

Table 7 presents the descriptive statistics for vinyl chloride concentration in upgradient groundwater based on 34 observations. The concentrations vary from 0.100 to 8.000, indicating a noticeable spread in the data, with a standard deviation of 1.952. The quartile values ($Q1 = 0.500$, median = 1.150, and $Q3 = 2.475$) show that most observations are concentrated at relatively low levels. The mean concentration (1.879) is higher than the median, suggesting a positively skewed distribution, which is further supported by the skewness value of 1.604. In addition, the kurtosis value of 5.005 indicates a leptokurtic pattern, reflecting heavier tails compared to the normal distribution. Overall, the data exhibit moderate variability with clear right-skewness and the presence of extreme values.

The XShanker distribution provides the ideal fit to the vinyl chloride concentration data among the 20 competing mixture distributions, as shown by the results in Table 8. This is because it yields the lowest AIC and BIC values. This suggests that the XShanker distribution achieves a superior balance between goodness-of-fit and model complexity compared to the alternative distributions. Consistently, the K-S test yields a statistic of 0.1109 with a corresponding p-value of 0.7975. No evidence exists to reject the null hypothesis as the p-value is considerably higher than the significance level (0.05). This indicates that the XShanker distribution supports a suitable fit to the data for vinyl chloride concentration.

Figure 8 provides a visual evaluation of the XShanker distribution for the vinyl chloride concentration data by showing the fitted pdf, obtained at $\hat{\theta} = 0.7567$ together with the observed histogram. The fitted curve represents the strong clustering of

Table 10. Predictive Probabilities and 95% Parametric Bootstrap CIs for Selected Vinyl Chloride Concentration Events

Event	Empirical Probability	Estimated Shanker Probability	95% Bootstrap CI	Absolute Error
$X \leq 0.5$ g/L	0.2941	0.2102	(0.1457, 0.2964)	0.0839
$X \leq 1.0$ g/L	0.4706	0.3872	(0.2843, 0.5116)	0.0834
$X \leq 2.0$ g/L	0.6765	0.6451	(0.5184, 0.7718)	0.0313
$X > 5.0$ g/L	0.1176	0.0575	(0.0200, 0.1210)	0.0601

Note: The 95% bootstrap CIs were obtained using 2,000 parametric bootstrap replications. Absolute error is the absolute difference between the empirical probability and the probability estimated from the fitted Shanker distribution

measurements near the lower end of the concentration range and accommodates the progressively sparse observations at higher levels. Its shape also follows the marked right asymmetry and the presence of several comparatively large concentration values. Together with the information criteria and goodness-of-fit findings in Table 8, this visual evidence further indicates that the XShanker distribution gives an appropriate representation of the dataset.

The P–P plot in Figure 9 shows that the XShanker distribution fits the vinyl chloride concentration data very well. The points on the graph are very close to the 45-degree reference line, which shows that the empirical cumulative distribution function (cdf) and the theoretical cdf of the XShanker distribution are very similar. There are some small differences in some areas, but they are not significant and do not follow any clear pattern. This shows that the XShanker distribution accurately reflects the data’s underlying behavior.

For the vinyl chloride data, the ML estimate was $\hat{\theta} = 0.7567$, and the sample size was $n = 34$. The likelihood and Wald CIs had similar widths of 0.3266 and 0.3247, respectively, whereas the bootstrap-t and BCa bootstrap CIs were wider, with widths of 0.3588 and 0.3609. The close agreement between the likelihood and Wald CIs is consistent with the simulation results, which showed similar AWs for these two methods across the parameter and sample-size settings. However, the ordering of the bootstrap interval widths differs from the average simulation pattern near $\theta = 0.75$ and $n = 30$, where the bootstrap CIs were slightly narrower. This difference may reflect the greater sample-to-sample variability of bootstrap procedures in small datasets, particularly for positively skewed observations with relatively large upper-tail values. Therefore, the vinyl chloride application provides partial rather than complete agreement with the simulation findings and reinforces the need for caution when applying bootstrap CIs to small samples.

After constructing the CIs for the XShanker parameter, the fitted model was further used to estimate probabilities for selected vinyl chloride concentration events. As shown in Table 10, the estimated probabilities were reasonably close to the corresponding empirical probabilities, with absolute errors ranging from 0.0313 to 0.0839. The closest agreement was obtained for concentrations not exceeding 2.0 g/L. Moreover, all empirical probabilities fell within their corresponding 95% parametric bootstrap CIs results provide additional support for

the practical usefulness of the fitted XShanker distribution in describing the vinyl chloride concentration data.

3.3 Limitations and Future Research

The findings should be interpreted within the parameter and sample-size ranges examined in the simulation. Values outside the investigated range and sample sizes below 10 were not considered; under such conditions, the ML estimator may be more skewed and numerical optimization or observed-information calculations may be less stable. In addition, the simulation assumed correct specification of the XShanker distribution, whereas robustness to contamination and model misspecification was not examined. The empirical applications were also limited to complete, uncensored datasets. Finally, because the XShanker distribution has a single parameter and an increasing hazard-rate form, it may not adequately represent data requiring independent control of scale and shape or decreasing, bathtub-shaped, or unimodal hazard functions. Future work may therefore investigate the interval estimation methods under more extreme parameter settings, model misspecification, and censored sampling schemes. Extensions to more flexible XShanker distributions are also of interest. The power XShanker distribution of Oha et al. (2024) introduces an additional power parameter, while the double XShanker distribution of Etaga et al. (2023b) modifies the mixture construction. CI estimation for these extensions, as well as regression analysis that relate the XShanker distribution parameter to explanatory variables through an appropriate link function, would broaden the scope of the methodology for lifetime, waiting-time, and environmental applications.

4. CONCLUSIONS

This study compared four 95% confidence interval (CI) methods for the XShanker distribution parameter: likelihood, Wald, bootstrap-t, and bias-corrected and accelerated (BCa) bootstrap intervals. Monte Carlo results showed that the likelihood and Wald CIs maintained empirical coverage rates closest to the nominal level across most parameter settings, particularly for small samples, whereas the bootstrap-t and BCa bootstrap CIs were generally narrower but exhibited undercoverage when the sample size was small. As the sample size increased, the coverage rates of all four methods became increasingly similar and their average widths decreased. Applications to bank cus-

tomers waiting times and vinyl chloride concentrations showed that the XShanker distribution provided a satisfactory fit and that the interval estimates were consistent with the simulation patterns.

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