

A Remark on the Energy Bounds and Degree-Based Topological Indices of the Zero-Divisor Graph $\mathbb{Z}_p[x]/\langle x^4 \rangle$

Vira Hari Krisnawati^{1*}, Ayunda Faizatul Musyarrofah¹, Ari Andari¹

¹Department of Mathematics, Faculty of Science, Technology, and Mathematics, Universitas Brawijaya, East Java, 65145, Indonesia

*Corresponding author: virahari@ub.ac.id

Abstract

The zero-divisor graph of a commutative ring, $\Gamma(\mathcal{R})$, is the prominent subject in graph theory. In previous research, there are results about the energy and topological indices of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$ for any prime p , but their properties contained errors in the characteristic polynomial and the associated energy bounds. While recent literature by Rather addressed these discrepancies using a vertex partitioning and quotient matrix approach, our work offers a distinct alternative. We employ a matrix-theoretic approach based on Schur complement to systematically derive the characteristic polynomial and its corresponding eigenvalues, thereby independently confirming and reinforcing the corrected energy bounds. Furthermore, this study significantly expands the literature beyond earlier works by investigating an extensive range of previously unexplored degree-based topological indices, including the second modified Zagreb, general and inverse general Randić, third and fifth symmetric division, harmonic, inverse sum, and forgotten indices.

Keywords

Graph Energy, Degree-Based Topological Indices, Zero-Divisor Graph

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1. INTRODUCTION

Graph theory, a field of discrete mathematics, examines the interactions between objects depicted as vertices and their connections represented as edges. In this work, we focus on simple, connected, and undirected graphs. Let $\mathbb{G}$ be a simple finite graph consisting of a set of vertices $\mathcal{V}(\mathbb{G}) = \{v_1, v_2, \dots, v_n\}$ and a set of edges $\mathcal{E}(\mathbb{G})$, with each edge connects a pair of distinct vertices. For $1 \leq i \neq j \leq n$, we write $v_i \sim v_j$ if they are adjacent and $v_i \not\sim v_j$ otherwise. The collection of vertices connected to a vertex v_i is referred to as its neighborhood, written as $N(v_i)$, and the degree of v_i corresponds to the number of these vertices, written as $d_{v_i} = |N(v_i)|$.

One of the standard representations of a graph is through its adjacency matrix, $A = (a_{ij})$, which is a matrix of order n , where the $a_{ij} = 1$ if $v_i \sim v_j$, and 0 otherwise. The spectral characteristics of a graph can be investigated through its adjacency matrix. Because this matrix is real and symmetric, it has real n eigenvalues and satisfy $\mu_1(\mathbb{G}) \geq \mu_2(\mathbb{G}) \geq \dots \geq \mu_n(\mathbb{G})$. The collection of these eigenvalues, counted with their algebraic multiplicities, is called the graph's spectrum, which can be expressed as a two-row matrix, where the first row lists the distinct eigenvalues and the second row indicates their corresponding multiplicities. This spectrum serves as the foundation for defining various spectral invariants. Among these invariants, one of the most

intensively studied is the energy of the graph, was originally put forward by Gutman (1978) in 1978 as

$$\text{En}(\mathbb{G}) = \sum_{i=1}^n |\mu_i|.$$

This concept originated in chemical graph theory as a tool to approximate the total π -electron energy of conjugated using molecular graphs (Graovac et al., 1976). Since then, graph energy has become a prominent topic in spectral graph theory for its mathematically interesting properties, as shown in recent literature (Baser and Ulucay, 2025; Jannat and Nath, 2026; Nalina et al., 2026; Zhang et al., 2025).

Another important class of graph invariants is known as topological indices. A topological index represents a quantitative value extracted from a graph's configuration, employed to measure its structural properties (Bokhary et al., 2022). These indices have proven useful for predicting physicochemical properties without the need for lengthy and costly experimental procedures (Mondal et al., 2023). Consequently, these descriptors are instrumental in minimizing the expenditure of resources, including time, equipment, and materials, within chemical studies. Topological indices also are categorized into degree-based and distance-based types. In this context, the former are particularly advantageous owing to their computational efficiency

and capacity to capture vital molecular attributes using solely vertex degrees. Various degree-based topological indices have been extensively studied, namely the first and second Zagreb (Gutman and Trinajstić, 2005), second modified Zagreb (Miličević et al., 2004), Narumi–Katayama (Narumi and Katayama, 1984), general Randić and inverse general Randić (Bollobás and Erdős, 1998), inverse sum and symmetric division (Vukićević and Gašperov, 2010), harmonic (Zhong, 2012), and forgotten (Furtula and Gutman, 2015). Moreover, many topological indices are based on the degree, see (Adnan et al., 2022; Cruz et al., 2024; Oboudi, 2023; Rauf et al., 2022; Wang et al., 2022).

In particular, studies on eigenvalues and topological indices of graphs constructed from algebraic structures have gained considerable attention. One of the most widely investigated examples is the graph first introduced by Beck (1988), namely the zero-divisor graph associated with a commutative ring $\mathcal{R}$, represented by $\Gamma(\mathcal{R})$. The graph is constructed with vertices representing all nonzero zero-divisors of $\mathcal{R}$, and edges connecting pairs whose product is zero (Anderson and Mulay, 2007). In 2011, Ahmadi and Jahani-Nezhad (2011) pioneered the investigation into the topological properties and energy associated with zero-divisor graphs. Furthermore, Singh and Bhat (2021) examined the adjacency matrix and the topological indices of $\Gamma(\mathbb{Z}_n)$ for $n \in \{p^2, p^m, pq, p^2q, pqr\}$. In 2021, Magi et al. (2020) identified the adjacency matrix and eigenvalues of $\Gamma(\mathbb{Z}_{p^2q^2})$. Additionally, Pirzada et al. (2021) obtained the eigenvalues of $\Gamma(\mathbb{Z}_{p^m q^n})$ for prime numbers $p < q$ with positive integers m, n and the Laplacian matrix of this graph is obtained by Rather et al. (2021). For further studies on the eigenvalues (spectrum) of the zero-divisor graph, we refer to (Ashraf et al., 2024; Bajaj and Panigrahi, 2022; Jamal and Imran, 2024; Rather et al., 2022; Rehman et al., 2024) and for topological indices, we refer to (Ahmad and López, 2021; Elahi et al., 2022; Koam et al., 2022; Rayer and Jeyaraj, 2023).

Recent developments in 2023 by Johnson and Sankar (2023) considered the energy of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$, but their results contained errors regarding the characteristic polynomial. In Rather (2026) addressed these discrepancies by employing a block matrix approach combined with graph-theoretic lemmas to identify the eigenvalues and their multiplicities.

In this paper, we revisit the characteristic polynomial of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$ to resolve the energy bound discrepancies in the work of Johnson and Sankar (2023). While Rather (2026) recently addressed these inaccuracies using vertex partitioning and a quotient matrix approach, our work provides a distinct, matrix-theoretic alternative. Specifically, we employ a Schur complement-based reduction technique to systematically derive the characteristic polynomial and its spectrum, reducing the main eigenvalue problem to an explicit cubic polynomial. This independent derivation corroborates and reinforces the corrected energy bounds reported by Rather. Furthermore, we significantly extend the literature beyond earlier studies by conducting a comprehensive analysis of various degree-based topological indices that remained unexplored in previous works

including the second modified Zagreb, general and inverse general Randić, third and fifth symmetric division, harmonic, inverse sum, and forgotten indices.

2. PRELIMINARIES

In this section, we shall restate fundamental concepts and definitions used in the next section. We begin by providing the matrix analysis tool, particularly block and circulant matrices. Then, we review several definitions of degree-based topological indices relevant to our study.

A block matrix is a matrix partitioned into smaller submatrices or blocks. This structure is particularly useful for simplifying computations such as determinant evaluation, especially in large matrices with repeated patterns. The followings lemma applies the Schur complement to efficiently compute the determinant of a block matrix.

Lemma 2.1. Zhang (2006) Let P, Q, R , and S be matrices of sizes $m \times m$, $m \times n$, $n \times m$, and $n \times n$, respectively. If P and S are invertible, then the determinant of the block matrix can be evaluated using the Schur complement as follows:

$$\det \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \det(P) \cdot \det(S - RP^{-1}Q) \\ = \det(S) \cdot \det(P - QS^{-1}R),$$

where $S - RP^{-1}Q$ is the Schur complement of block P , and $P - QS^{-1}R$ is the Schur complement of block S .

Next, we consider a special class of matrices known as circulant matrices. A circulant matrix of order $n \times n$ with entries $c_0, c_1 \in \mathbb{R}$ is denoted by $C_{(c_0, c_1, n)}$ and is defined by the following structure:

$$C_{(c_0, c_1, n)} = \begin{pmatrix} c_0 & c_1 & c_1 & \dots & c_1 \\ c_1 & c_0 & c_1 & \dots & c_1 \\ c_1 & c_1 & c_0 & \dots & c_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_1 & c_1 & \dots & c_0 \end{pmatrix}_{n \times n}.$$

The following propositions provide the determinant and inverse of such matrices.

Proposition 2.2. Magi et al. (2020) Consider a circulant matrix $C_{(c_0, c_1, n)}$. The determinant of C is evaluated as $\det(C_{(c_0, c_1, n)}) = (c_0 + (n-1)c_1)(c_0 - c_1)^{n-1}$.

Proposition 2.3. Magi et al. (2020) For any nonsingular circulant matrix $C_{(c_0, c_1, n)}$, the inverse of $C_{(c_0, c_1, n)}$ is

$$C_{(c_0, c_1, n)}^{-1} = \frac{1}{\det(C_{(c_0, c_1, n)})} \times \begin{bmatrix} \alpha_{n-1} & \beta_{n-1} & \beta_{n-1} & \dots & \beta_{n-1} \\ \beta_{n-1} & \alpha_{n-1} & \beta_{n-1} & \dots & \beta_{n-1} \\ \beta_{n-1} & \beta_{n-1} & \alpha_{n-1} & \dots & \beta_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{n-1} & \beta_{n-1} & \beta_{n-1} & \dots & \alpha_{n-1} \end{bmatrix},$$

$$C_{(c_0, c_1, n)}^{-1} = \frac{1}{\det(C_{(c_0, c_1, n)})} C_{(\alpha_{n-1}, \beta_{n-1}, n)},$$

where $\alpha_{n-1} = (c_0 + c_1(n-2))(c_0 - c_1)^{n-2}$ and $\beta_{n-1} = -c_1(c_0 - c_1)^{n-2}$.

To support the spectral analysis, particularly in establishing bounds for graph energy, we utilize the following inequality involving symmetric means:

Proposition 2.4. Biler and Witkowski (2017) Given the set $\{f_1, f_2, f_3, \dots, f_s\}$, with $f_i \in \mathbb{R}^+$. Let Π_k be defined as follows:

$$\begin{aligned}\Pi_1 &= \frac{f_1 + f_2 + f_3 + \dots + f_s}{s}, \\ \Pi_2 &= \frac{1}{\frac{s(s-1)}{2}}(f_1f_2 + f_1f_3 + \dots + f_1f_s + f_2f_3 + \dots + f_{s-1}f_s), \\ &\vdots \\ \Pi_s &= f_1f_2f_3 \dots f_s.\end{aligned}$$

It follows from the Maclaurin symmetric mean inequality:

$$\Pi_1^{1/s} \geq \Pi_2^{1/(s-1)} \geq \Pi_3^{1/(s-2)} \geq \dots \geq \Pi_s^{1/1} \quad (1)$$

where equality is achieved if and only if $f_1 = f_2 = f_3 = \dots = f_s$.

We now recall the definitions of various degree-based topological indices of the graph $\mathbb{G}$ used in this study, including the second modified Zagreb ${}^mM_2(\mathbb{G})$ (Miličević et al., 2004), general Randić $R_\alpha(\mathbb{G})$, $\alpha \in \mathbb{R}^+$ (Bollobás and Erdős, 1998), inverse general Randić $RR_\alpha(\mathbb{G})$, $\alpha \in \mathbb{R}^+$ (Bollobás and Erdős, 1998), third symmetric divisions $SSD_3(\mathbb{G})$ (Alali et al., 2023), fifth symmetric divisions $SSD_5(\mathbb{G})$ (Alali et al., 2023), harmonic $H(\mathbb{G})$ (Zhong, 2012), inverse sum $I(\mathbb{G})$ (Vukičević and Gašperov, 2010), and forgotten topological index $F(\mathbb{G})$ (Furtula and Gutman, 2015):

$$\begin{aligned}{}^mM_2(\mathbb{G}) &= \sum_{v_i v_j \in \mathcal{E}(\mathbb{G})} \left(\frac{1}{d_{v_i} \cdot d_{v_j}} \right), \\ R_\alpha(\mathbb{G}) &= \sum_{v_i v_j \in \mathcal{E}(\mathbb{G})} \left((d_{v_i} \cdot d_{v_j})^\alpha \right), \quad \alpha \in \mathbb{R}^+, \\ RR_\alpha(\mathbb{G}) &= \sum_{v_i v_j \in \mathcal{E}(\mathbb{G})} \left(\frac{1}{(d_{v_i} \cdot d_{v_j})^\alpha} \right), \quad \alpha \in \mathbb{R}^+, \\ SSD_3(\mathbb{G}) &= \sum_{v_i v_j \in \mathcal{E}(\mathbb{G})} \left(d_{v_i} \cdot d_{v_j} (d_{v_i} + d_{v_j}) \right),\end{aligned}$$

$$SSD_5(\mathbb{G}) = \sum_{v_i v_j \in \mathcal{E}(\mathbb{G})} \left(\frac{d_{v_i}}{d_{v_j}} + \frac{d_{v_j}}{d_{v_i}} \right),$$

$$H(\mathbb{G}) = \sum_{v_i v_j \in \mathcal{E}(\mathbb{G})} \left(\frac{2}{d_{v_i} + d_{v_j}} \right),$$

$$I(\mathbb{G}) = \sum_{v_i v_j \in \mathcal{E}(\mathbb{G})} \left(\frac{d_{v_i} \cdot d_{v_j}}{d_{v_i} + d_{v_j}} \right),$$

$$F(\mathbb{G}) = \sum_{v_i v_j \in \mathcal{E}(\mathbb{G})} \left(d_{v_i}^2 + d_{v_j}^2 \right).$$

3. RESULTS AND DISCUSSIONS

3.1 Energy of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$

The graph $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$ is constructed from the quotient ring $\mathbb{Z}_p[x]/\langle x^4 \rangle$, whose elements are of the form $ax^3 + bx^2 + cx + d + \langle x^4 \rangle$, where $a, b, c, d \in \mathbb{Z}_p$. In Johnson and Sankar (2023) constructed this graph for prime p , consisting of $p^3 - 1$ vertices corresponding to the nonzero zero-divisors, and $\frac{1}{2}(3p^4 - 4p^3 - p^2 + 2)$ edges. For completeness, we briefly describe the structure of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$. The set of vertices of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$ are able to be partitioned as

$$\mathcal{A} = \{kx^3 \mid k \in \mathbb{Z}_p \setminus \{0\}\}, \quad (2)$$

$$\mathcal{B} = \{lx^3 + kx^2 \mid l \in \mathbb{Z}_p, k \in \mathbb{Z}_p \setminus \{0\}\}, \quad (3)$$

$$\mathcal{C} = \{mx^3 + lx^2 + kx \mid l, m \in \mathbb{Z}_p, k \in \mathbb{Z}_p \setminus \{0\}\}. \quad (4)$$

The cardinality of $\mathcal{A}$ is $p - 1$. Analogously, we derive $|\mathcal{B}| = p(p - 1)$ and $|\mathcal{C}| = p^2(p - 1)$. To provide a more detailed characterization of the structure of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$, we examine how adjacency is defined among the elements of the vertex sets $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. The adjacency is determined as follows:

1. Adjacency Within $\mathcal{A}$:

If $k_ix^3, k_jx^3 \in \mathcal{A}$. Then, $(k_ix^3)(k_jx^3) = k_ik_jx^6 \equiv 0 \pmod{x^4}$. This result confirms that any two distinct vertices in $\mathcal{A}$ are adjacent to each other. Furthermore, for $k_ix^3 \in \mathcal{A}$ and $l_ix^3 + k_jx^2 \in \mathcal{B}$, we have $(k_ix^3)(l_ix^3 + k_jx^2) = k_ili^6 + k_ik_jx^5 \equiv 0 \pmod{x^4}$. Hence, every vertex in $\mathcal{A}$ is adjacent to every vertex in $\mathcal{B}$. Similarly, for $k_ix^3 \in \mathcal{A}$ and $m_ix^3 + l_ix^2 + k_jx \in \mathcal{C}$, we obtain $(k_ix^3)(m_ix^3 + l_ix^2 + k_jx) = k_im_ix^6 + k_ili^5 + k_ik_jx^4 \equiv 0 \pmod{x^4}$. Thus, each vertex in $\mathcal{A}$ is also adjacent to every vertex in $\mathcal{C}$.

2. Adjacency Within $\mathcal{B}$:

If $l_ix^3 + k_ix^2$ and $l_jx^3 + k_jx^2$ in $\mathcal{B}$. Their product yields

$$\begin{aligned}(l_ix^3 + k_ix^2)(l_jx^3 + k_jx^2) &= l_il_jx^6 + (l_ik_j + k_il_j)x^5 \\ &\quad + k_ik_jx^4 \equiv 0 \pmod{x^4},\end{aligned}$$

indicating that all distinct pairs of vertices in $\mathcal{B}$ are adjacent. However, if $l_ix^3 + k_ix^2 \in \mathcal{B}$ and $m_jx^3 + l_jx^2 + k_jx \in \mathcal{C}$. Then,

$$\begin{aligned}(l_ix^3 + k_ix^2)(m_jx^3 + l_jx^2 + k_jx) &= l_im_jx^6 + (l_il_j + k_im_j)x^5 \\ &\quad + (l_ik_j + k_il_j)x^4 + k_ik_jx^3 \not\equiv 0 \pmod{x^4}.\end{aligned}$$

Therefore, the sets $\mathcal{B}$ and $\mathcal{C}$ are mutually nonadjacent.

3. Adjacency Within $\mathcal{C}$:

If $m_i x^3 + l_i x^2 + k_i x$ and $m_j x^3 + l_j x^2 + k_j x$ in $\mathcal{C}$. Their product is given by

$$\begin{aligned} & (m_i x^3 + l_i x^2 + k_i x)(m_j x^3 + l_j x^2 + k_j x) \\ &= m_i m_j x^6 + (m_i l_j + l_i m_j) x^5 \\ &+ (m_i k_j + l_i l_j + k_i m_j) x^4 \\ &+ (l_i k_j + k_i l_j) x^3 + k_i k_j x^2 \\ &\not\equiv 0 \pmod{x^4}. \end{aligned}$$

Consequently, the vertices in $\mathcal{C}$ are pairwise nonadjacent. Therefore, the adjacency matrix of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$ is as follows:

$$A(\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)) = \begin{matrix} & \mathcal{C} & \mathcal{B} & \mathcal{A} \\ \begin{matrix} \mathcal{C} \\ \mathcal{B} \\ \mathcal{A} \end{matrix} & \begin{pmatrix} \mathbb{O}_{p^3-p^2} & \mathbb{O}_{(p^3-p^2) \times (p^2-p)} & \mathcal{J}_{(p^3-p^2) \times (p-1)} \\ \mathbb{O}_{(p^2-p) \times (p^3-p^2)} & (\mathcal{J} - I)_{p^2-p} & \mathcal{J}_{(p^2-p) \times (p-1)} \\ \mathcal{J}_{(p-1) \times (p^3-p^2)} & \mathcal{J}_{(p-1) \times (p^2-p)} & (\mathcal{J} - I)_{p-1} \end{pmatrix} \end{matrix} \quad (5)$$

Throughout this paper, the symbols $\mathbb{O}$, $\mathcal{J}$, and I correspond to the zero matrix, the all-ones matrix, and the identity matrix, respectively.

We shall now examine the structural energy behavior of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$. Ahead of this, we first recall the relevant properties in (Johnson and Sankar, 2023).

Theorem 3.1. (Theorem 3.2, (Johnson and Sankar, 2023)). If $\mathcal{R} = \mathbb{Z}_p[x]/\langle x^4 \rangle$ with prime p , then the energy of $\Gamma(\mathcal{R})$ satisfies

$$\text{En}(\Gamma(\mathcal{R})) \geq \frac{1}{2}(p^5 + p^4 - 4p^3 + 3p^2 - 5p + 4).$$

A crucial step in the proof of **Theorem 3.1**, involves the characteristic polynomial associated with $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$ is written as follows:

$$\begin{aligned} & (\lambda - p + 1)^{p^2(p-1)-1} (\lambda - p^3 + 1)^{(p-2)} \\ & \times \left(\lambda - \frac{1}{2}[p^4 - 2p^3 + 3p] \right)^{p(p-1)-1} \phi(\lambda) = 0, \end{aligned} \quad (6)$$

where $\phi(\lambda)$ is

$$\begin{aligned} \phi(\lambda) &= \lambda^3 - \frac{1}{2}(p^4 - 2p^2 + 5p - 2)\lambda^2 \\ &+ \frac{1}{2}(p^2 - p)(p^2 + p + 1)(p^3 - 2p^2 - 2p + 5)\lambda \\ &- \frac{1}{2}p(p-1)^2(p^4 - 2p^3 - 2p^2 + 3p + 2). \end{aligned}$$

However, we find a mistake in **Theorem 3.1**. For $p = 2$, using Equation (5), we find that the characteristic polynomial of $\Gamma(\mathbb{Z}_2[x]/\langle x^4 \rangle)$ is

$$\lambda(\lambda - 1)^4(\lambda - 3)(\lambda - 7) = 0,$$

which gives the spectrum

$$\text{Spec} \left(A \left(\Gamma(\mathbb{Z}_2[x]/\langle x^4 \rangle) \right) \right) = \begin{pmatrix} 0 & 1 & 3 & 7 \\ 1 & 4 & 1 & 1 \end{pmatrix}. \quad (7)$$

Meanwhile, if we construct the $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$ for $p = 2$, we have the vertices are

$$\begin{aligned} \mathcal{A} &= \{x^3\}, \\ \mathcal{B} &= \{x^2, x^3 + x^2\}, \\ \mathcal{C} &= \{x, x^3 + x^2 + x, x^3 + x, x^2 + x\}. \end{aligned}$$

The structure of the graph is depicted in Figure 1. We express the adjacency matrix of $\Gamma(\mathbb{Z}_2[x]/\langle x^4 \rangle)$ as follows:

$$A(\Gamma(\mathbb{Z}_2[x]/\langle x^4 \rangle)) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Using Wolfram Mathematica, the characteristic polynomial of $A(\Gamma(\mathbb{Z}_2[x]/\langle x^4 \rangle))$ is found to be

$$(-\lambda)^3(-\lambda - 1)(\lambda^3 - \lambda^2 - 6\lambda + 4) = 0.$$

Hence, the spectrum is

$$\text{Spec} \left(A \left(\Gamma(\mathbb{Z}_2[x]/\langle x^4 \rangle) \right) \right) = \begin{pmatrix} 0 & -1 & -2.3234 & 0.6421 & 2.6813 \\ 3 & 1 & 1 & 1 & 1 \end{pmatrix} \quad (8)$$

From this observation, we identify a mismatch between the spectrum of $\Gamma(\mathbb{Z}_2[x]/\langle x^4 \rangle)$ between Equation (7) and the spectrum given in Equation (8). Specifically, used Equation (6), yields a different set of eigenvalues to those obtained through explicit computation for $p = 2$.

This inconsistency implies that the characteristic polynomial stated in **Theorem 3.1** is not valid when $p = 2$. Consequently, since the energy bound in **Theorem 3.1** is derived based on this incorrect polynomial, the claimed lower bound of $\text{En}(\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle))$ is also unreliable for $p = 2$. This finding suggests that further verification or correction is needed in the formulation of the characteristic polynomial and the derived energy result in (Johnson and Sankar, 2023).

The following theorem characterizes the eigenvalues associated with $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$.

Theorem 3.2. If $\mathcal{R} = \mathbb{Z}_p[x]/\langle x^4 \rangle$, then the following statements describe the eigenvalues of $\Gamma(\mathcal{R})$.

1. The eigenvalues of $\Gamma(\mathcal{R})$ are 0, with multiplicity $p^3 - p^2 - 1$, and -1 , with multiplicity $p^2 - 3$.
2. The rest of the eigenvalues of $\Gamma(\mathcal{R})$ are the roots of the cubic polynomial given by

$$\lambda^3 - (p^2 - 3)\lambda^2 - (p^4 - 2p^3 + 2p^2 - 2)\lambda + p^2(p-1)^2(p^2 - p - 1) = 0.$$

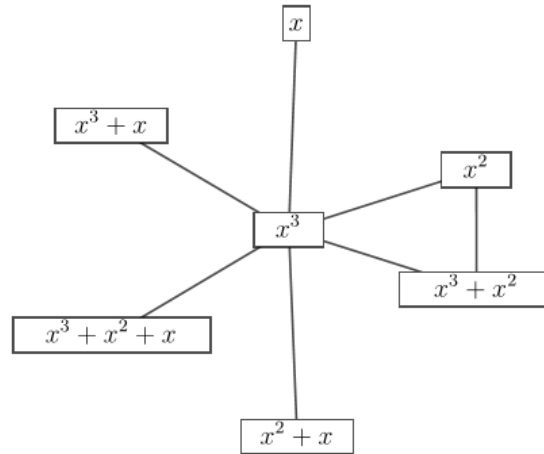


Figure 1. Zero-Divisor Graph of $\mathbb{Z}_2[x]/\langle x^4 \rangle$

Proof. Let $\mathcal{R} = \mathbb{Z}_p[x]/\langle x^4 \rangle$ and the adjacency matrix of $\mathcal{R}$ given by Equation (5) be denoted by $\mathcal{M}$. Then, the eigenvalues of $\Gamma(\mathcal{R})$ are given by $|\mathcal{M} - \lambda I| = 0$. Thus,

$$\mathcal{M} - \lambda I = \begin{pmatrix} \mathcal{C}_{(-\lambda, 0, p^3-p^2)} & \mathcal{C}_{(p^3-p^2) \times (p^2-p)} & \mathcal{F}_{(p^3-p^2) \times (p-1)} \\ \mathcal{C}_{(p^2-p) \times (p^3-p^2)} & \mathcal{C}_{(-\lambda, -1, p^2-p)} & \mathcal{F}_{(p^2-p) \times (p-1)} \\ \mathcal{F}_{(p-1) \times (p^3-p^2)} & \mathcal{F}_{(p-1) \times (p^2-p)} & \mathcal{C}_{(-\lambda, -1, p-1)} \end{pmatrix} = \begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix}$$

Note that $P = \mathcal{C}_{(-\lambda, 0, p^3-p^2)}$ is a scalar matrix:

$$P = \begin{pmatrix} -\lambda & 0 & 0 & \dots & 0 \\ 0 & -\lambda & 0 & \dots & 0 \\ 0 & 0 & -\lambda & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -\lambda \end{pmatrix}_{p^3-p^2}$$

If $\lambda \neq 0$, then P is invertible. Thus, by Lemma 2.1, we have

$$\det(\mathcal{M} - \lambda I) = \det(P) \cdot \det(R - Q^T P^{-1} Q). \quad (9)$$

Consequently, we get

$$\det(P) = (-\lambda)^{p^3-p^2} \quad (10)$$

Also $P^{-1} = -\frac{1}{\lambda} I$, where I is identity matrix of size (p^3-p^2) . Thus, $R - Q^T P^{-1} Q$ becomes

$$R - Q^T P^{-1} Q = \begin{pmatrix} \mathcal{C}_{(-\lambda, 1, p^2-p)} & \mathcal{F}_{(p^2-p) \times (p-1)} \\ \mathcal{F}_{(p-1) \times (p^2-p)} & \mathcal{C}_{(-\lambda + \frac{p^3-p^2}{\lambda}, 1 + \frac{p^3-p^2}{\lambda}, p-1)} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

By Proposition 2.2, the determinant of A_{11} is given by

$$\det(A_{11}) = h(\lambda)(-\lambda - 1)^{p^2-p-1}, \quad (11)$$

where $h(\lambda) = -\lambda + p^2 - p - 1$.

For $\lambda \neq -1$ and $\lambda \neq p^2 - p - 1$, we have $\det(A_{11}) \neq 0$, which implies that A_{11} is invertible. Thus, applying Lemma 2.1 once again, we obtain

$$\det(R - Q^T P^{-1} Q) = \det(A_{11}) \cdot \det(A_{22} - A_{21} A_{11}^{-1} A_{12}). \quad (12)$$

The inverse of A_{11} , by Proposition 2.3, is given by

$$A_{11}^{-1} = \frac{1}{h(\lambda)(\lambda + 1)} C_{(\lambda - p^2 + p + 2, 1, p^2 - p)}.$$

Substituting into the Schur complement, the resulting matrix becomes

$$A_{22} - A_{21} A_{11}^{-1} A_{12} = C_{(-\lambda + \frac{p^3-p^2}{\lambda} - \frac{p^2-p}{h(\lambda)}, 1 + \frac{p^3-p^2}{\lambda} - \frac{p^2-p}{h(\lambda)}, p-1)}.$$

Consequently, the determinant of this matrix is

$$(-1)^{p-1} (\lambda + 1)^{p-2} \left(\frac{\lambda^2 h(\lambda) - (p-2) \lambda h(\lambda) + p(p-1)^2 \lambda - p^2 (p-1)^2 h(\lambda)}{\lambda h(\lambda)} \right). \quad (13)$$

Using Equations (10), (11), (12), and (13) in Equation (9), the characteristic polynomial associated with $\Gamma(\mathcal{R})$ is obtained as

$$(\lambda)^{p^3-p^2-1} (\lambda + 1)^{p^2-3} \phi(\lambda) = 0$$

where $\phi(\lambda) = \lambda^3 - (p^2 - 3)\lambda^2 - (p^4 - 2p^3 + 2p^2 - 2)\lambda + p^2(p - 1)^2(p^2 - p - 1)$.

Hence $\lambda = 0$ and $\lambda = -1$ are eigenvalues of $\Gamma(\mathcal{R})$ with multiplicities $p^3 - p^2 - 1$ and $p^2 - 3$ respectively. Also if $\lambda \neq 0$ and $\lambda \neq -1$, then λ satisfies

$$\phi(\lambda) = \lambda^3 - (p^2 - 3)\lambda^2 - (p^4 - 2p^3 + 2p^2 - 2)\lambda + p^2(p - 1)^2(p^2 - p - 1) = 0.$$

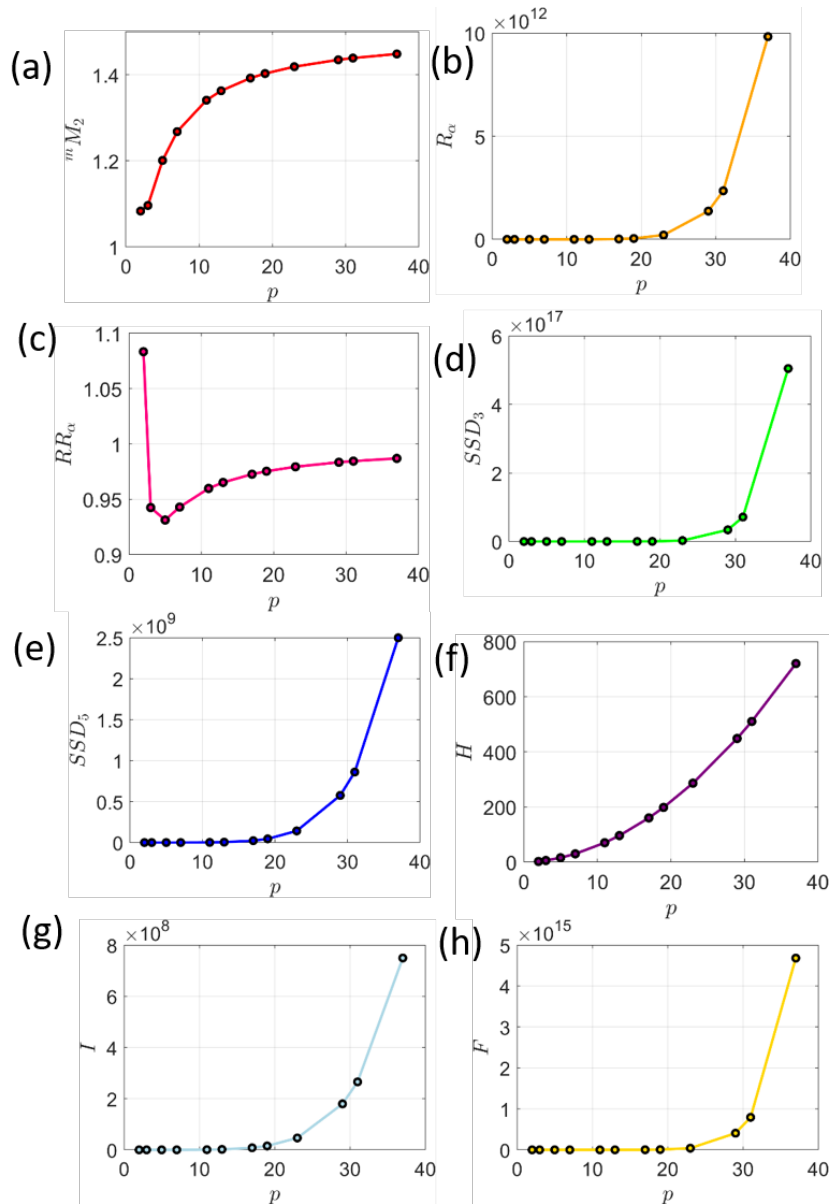


Figure 2. The Two-Dimensional Plot of (a) Second Modified Zagreb Index mM_2 of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$, (b) General Randić Index R_1 of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$, (c) Inverse General Randić Index RR_1 of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$, (d) Third Symmetric Division Index SSD_3 of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$, (e) Fifth Symmetric Division Index SSD_5 of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$, (f) Harmonic Index H of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$, (g) Inverse Sum Index I of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$, and (h) Forgotten Index F of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$

Theorem 3.3. If $\mathcal{R} = \mathbb{Z}_p[x]/\langle x^4 \rangle$, where any prime p , then

(14)

$$\text{En}(\Gamma(\mathcal{R})) \leq p^2 - 3 + \sqrt{9p^4 - 12p^3 - 6p^2 + 15}$$

and

$$\text{En}(\Gamma(\mathcal{R})) \geq p^2 - 3 + \sqrt{3p^4 - 4p^3 - 2p^2 + 5 + 6(p^6 - 3p^5 + 2p^4 + p^3 - p^2)^{2/3}}.$$

Proof. Let $\mathcal{R} = \mathbb{Z}_p[x]/\langle x^4 \rangle$ and $\zeta_1, \zeta_2, \zeta_3$ be the eigenvalues obtained from Equation (14).

$$\lambda^3 - (p^2 - 3)\lambda^2 - (p^4 - 2p^3 + 2p^2 - 2)\lambda + p^2(p - 1)^2(p^2 - p - 1) = 0.$$

Their sum, pairwise products, and total product are as follows:

$$\sum_{i=1}^3 \zeta_i = -p^2 + 3, \quad (15)$$

$$\sum_{1 \leq i < j \leq 3} \zeta_i \zeta_j = -p^4 + 2p^3 - 2p^2 + 2, \quad (16)$$

$$\prod_{i=1}^3 \zeta_i = p^2(p^2 - p - 1)(p - 1)^2. \quad (17)$$

Furthermore, the energy of $\Gamma(\mathcal{R})$ can be expressed as the following:

$$\begin{aligned} \text{En}(G) &= \sum_{i=1}^{p^3-1} |\lambda_i| \\ &= (p^3 - p^2 - 1)|0| + (p^2 - 3)|-1| + |\zeta_1| + |\zeta_2| + |\zeta_3| \\ &= p^2 - 3 + |\zeta_1| + |\zeta_2| + |\zeta_3|. \end{aligned} \quad (18)$$

In order to estimate the sum $|\zeta_1| + |\zeta_2| + |\zeta_3|$, we apply the Maclaurin symmetric mean inequality in Equation (1), specifically for the case $\Pi_1 \geq (\Pi_2)^{1/2}$. Applying this inequality to the set $\{|\zeta_1|, |\zeta_2|, |\zeta_3|\}$, we have

$$\left(\frac{|\zeta_1| + |\zeta_2| + |\zeta_3|}{3} \right)^2 \geq \frac{1}{\frac{3(3-1)}{2}} (|\zeta_1 \zeta_2| + |\zeta_1 \zeta_3| + |\zeta_2 \zeta_3|),$$

which simplifies to

$$\left(\sum_{i=1}^3 |\zeta_i| \right)^2 \geq 3 \left(\sum_{1 \leq i < j \leq 3} |\zeta_i| |\zeta_j| \right). \quad (19)$$

We now observe that

$$\left(\sum_{i=1}^3 |\zeta_i| \right)^2 = \sum_{i=1}^3 |\zeta_i|^2 + 2 \left(\sum_{1 \leq i < j \leq 3} |\zeta_i| |\zeta_j| \right).$$

Thus, we can express the pairwise products as

$$\sum_{1 \leq i < j \leq 3} |\zeta_i| |\zeta_j| = \frac{1}{2} \left(\left(\sum_{i=1}^3 |\zeta_i| \right)^2 - \sum_{i=1}^3 \zeta_i^2 \right). \quad (20)$$

We get the following inequality, by substituting Equation (20) into Equation (19),

$$\left(\sum_{i=1}^3 |\zeta_i| \right)^2 \leq 3 \sum_{i=1}^3 \zeta_i^2. \quad (21)$$

Next, we expand the right-hand side of Equation (21) as follows:

$$\left(\sum_{i=1}^3 |\zeta_i| \right)^2 \leq 3 \left(\left(\sum_{i=1}^3 \zeta_i \right)^2 - 2 \sum_{1 \leq i < j \leq 3} \zeta_i \zeta_j \right). \quad (22)$$

Substituting Equations (15) and (16) into Equation (17), we get

$$\sum_{i=1}^3 |\zeta_i| \leq \sqrt{9p^4 - 2p^3 - 6p^2 + 15}.$$

Based on Equation (18), the upper bound for the energy of G is

$$p^2 - 3 + \sqrt{9p^4 - 2p^3 - 6p^2 + 15}.$$

Next, we calculate the lower bound of $\text{En}(\Gamma(\mathcal{R}))$. Using the second inequality of the Maclaurin symmetric mean in Equation (1), we have

$$\frac{1}{\frac{3(3-1)}{2}} (|\zeta_1| |\zeta_2| + |\zeta_1| |\zeta_3| + |\zeta_2| |\zeta_3|) \geq (|\zeta_1| |\zeta_2| |\zeta_3|)^{2/3}.$$

Thus,

$$\frac{1}{3} \left(\sum_{1 \leq i < j \leq 3} |\zeta_i| |\zeta_j| \right) \geq \left(\prod_{i=1}^3 |\zeta_i| \right)^{2/3}. \quad (23)$$

Inserting Equation (20) into Equation (23), yields

$$\left(\sum_{i=1}^3 |\zeta_i| \right)^2 \geq \sum_{i=1}^3 \zeta_i^2 + 6 \left(\prod_{i=1}^3 |\zeta_i| \right)^{2/3}.$$

Upon expansion, we express it as

$$\left(\sum_{i=1}^3 |\zeta_i| \right)^2 \geq \left(\sum_{i=1}^3 \zeta_i \right)^2 - 2 \left(\sum_{1 \leq i < j \leq 3} \zeta_i \zeta_j \right) + 6 \left(\prod_{i=1}^3 |\zeta_i| \right)^{2/3}. \quad (24)$$

Replacing Equations (15), (16), and (17) into Equation (24), obtain

$$\sum_{i=1}^3 |\zeta_i| \geq \sqrt{3p^4 - 4p^3 - 2p^2 + 5 + 6(p^6 - 3p^5 + 2p^4 + p^3 - p^2)^{2/3}}.$$

Thus, based on Equation (18), the lower bound for the energy of $\Gamma(\mathcal{R})$ is

$$p^2 - 3 + \sqrt{3p^4 - 4p^3 - 2p^2 + 5 + 6(p^6 - 3p^5 + 2p^4 + p^3 - p^2)^{2/3}}.$$

To clarify the relationships regarding the calculated lower and upper bounds of the graph energy and the actual energy values obtained from the characteristic polynomial, Table 1 presents the numerical results for the prime numbers $2 \leq p \leq 37$.

From Table 1, it is evident that for all primes considered, the computed energy values lie very near to the derived lower and upper bounds. In particular, the differences between the bounds and the actual energy values are small even for larger primes. Moreover, the actual energy values are closer to the lower bound than to the upper bound. This indicates that the formulas in **Theorem 3.3** provide a tight and reliable estimate of $\text{En}(\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle))$. This demonstrates that the techniques used to derive these bounds work well for different prime p .

Table 1. Graph Energy of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$ for Several Primes p

p	Lower Bound	Graph Energy	Upper Bound
2	6.3027	6.6468	7.2450
3	23.6997	24.0869	25.1311
5	82.4096	83.0863	85.1664
7	173.0957	174.0936	177.2021
11	450.4475	452.1031	457.2433
13	637.1192	639.1060	645.2557
17	1106.4592	1109.1101	1117.2725
19	1389.1282	1392.1115	1401.2785
23	2050.4649	2054.1137	2065.2875
29	3282.4683	3287.1160	3301.2966
31	3757.1358	3762.1165	3777.2988
37	5373.1378	5379.1179	5397.3043

3.2 Degree-Based Topological Indices of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$

This part focuses on determining various degree-based indices of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$. In particular, we evaluate the second modified Zagreb index, the general Randić index, along with its inverse, and the third and fifth symmetric division indices. Furthermore, our analysis covers the harmonic index, the inverse sum index, and the forgotten index. **Lemma 4.1.** If $\mathcal{R} \cong \mathbb{Z}_p[x]/\langle x^4 \rangle$, where any prime p , then

$$d_u = \begin{cases} p^3 - 2, & \text{if } u \in \mathcal{A}, \\ p^2 - 2, & \text{if } u \in \mathcal{B}, \\ p - 1, & \text{if } u \in \mathcal{C}, \end{cases}$$

where d_u are the degrees of $\Gamma(\mathcal{R})$.

Proof. Suppose that $\mathcal{R} = \mathbb{Z}_p[x]/\langle x^4 \rangle$, where p is a prime. From the adjacency matrix given in Equation (5), it can be observed that the vertices in $\mathcal{A}$ are adjacent to all other vertices in $\Gamma(\mathcal{R})$. Consequently, for any element $u \in \mathcal{A}$ its degree can be formulated as follows:

$$d_u = |\mathcal{A}| - 1 + |\mathcal{B}| + |\mathcal{C}| = p^3 - 2.$$

In contrast, each vertex in $\mathcal{B}$ is adjacent only to the vertices in $\mathcal{A}$ and to other vertices in $\mathcal{B}$. Accordingly, the degree computation for $u \in \mathcal{B}$ yields:

$$d_u = |\mathcal{A}| + |\mathcal{B}| - 1 = p^2 - 2.$$

Meanwhile, the vertices in $\mathcal{C}$ are adjacent only to those in $\mathcal{A}$, so the degree of $u \in \mathcal{C}$ is

$$d_u = |\mathcal{A}| = p - 1.$$

To compute the degree-based topological indices of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$, we partition the edge set $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$ into four disjoint subsets $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$, and $\mathcal{E}_4$ based on the matrix adjacency of this graph in Equation (5) and their degrees as established in Lemma 4.1.

$$\mathcal{E}_1 = \{v_i v_j \in \mathcal{E}(\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)) : d_{v_i} = p^3 - 2, d_{v_j} = p^3 - 2, v_i, v_j \in \mathcal{A}\}$$

$$\mathcal{E}_2 = \{v_i v_j \in \mathcal{E}(\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)) : d_{v_i} = p^3 - 2, d_{v_j} = p^2 - 2, v_i \in \mathcal{A}, v_j \in \mathcal{B}\}$$

$$\mathcal{E}_3 = \{v_i v_j \in \mathcal{E}(\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)) : d_{v_i} = p^3 - 2, d_{v_j} = p - 1, v_i \in \mathcal{A}, v_j \in \mathcal{C}\}$$

$$\mathcal{E}_4 = \{v_i v_j \in \mathcal{E}(\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)) : d_{v_i} = p^2 - 2, d_{v_j} = p^2 - 2, v_i, v_j \in \mathcal{B}\}$$

where $|\mathcal{E}_1| = \frac{1}{2}(p^2 - 3p + 2)$, $|\mathcal{E}_2| = p^3 - 2p^2 + p$, $|\mathcal{E}_3| = p^4 - 2p^3 + p$, and $|\mathcal{E}_4| = \frac{1}{2}(p^4 - 2p^3 + p)$.

Theorem 4.2. If $\mathcal{R} = \mathbb{Z}_p[x]/\langle x^4 \rangle$, with any prime p , then

$${}^m M_2(\Gamma(\mathcal{R})) = \frac{(p-1)(3p^9 - p^8 - 7p^7 - 10p^6 + 9p^5 + 18p^4 + 4p^3 - 4p^2 - 8p - 8)}{2(p^2 - 2)^2(p^3 - 2)^2}$$

$$R_\alpha(\Gamma(\mathcal{R})) = \frac{1}{2}(p-1) \left[(p-2)(p^3 - 2)^{2\alpha} + p(p^2 - p - 1)(p^2 - 2)^{2\alpha} \right] + p(p-1)^2(p^3 - 2)^\alpha \left((p^2 - 2)^\alpha + p(p-1)^\alpha \right)$$

$$RR_\alpha(\Gamma(\mathcal{R})) = \frac{1}{2} \left(\frac{p-2}{(p^3 - 2)^{2\alpha}} + \frac{p(p^2 - p - 1)}{(p^2 - 2)^{2\alpha}} \right) + \frac{p(p-1)^2}{(p^3 - 2)^\alpha} \left(\frac{1}{(p^2 - 2)^\alpha} + \frac{p}{(p-1)^\alpha} \right)$$

for $\alpha \in \mathbb{R}^+$.

Proof. Let $\mathcal{R} = \mathbb{Z}_p[x]/\langle x^4 \rangle$, where p is prime. By the definition of the second modified Zagreb index and the edge partition given above, we have the modified second Zagreb index of $\Gamma(\mathcal{R})$ is

$$\begin{aligned} {}^m M_2(\Gamma(\mathcal{R})) &= \sum_{v_i v_j \in \mathcal{E}(\Gamma(\mathcal{R}))} \left(\frac{1}{d_{v_i} \cdot d_{v_j}} \right) \\ &= |\mathcal{E}_1| \left(\frac{1}{d_{v_i} \cdot d_{v_j}} \right)_{v_i v_j \in \mathcal{E}_1} + |\mathcal{E}_2| \left(\frac{1}{d_{v_i} \cdot d_{v_j}} \right)_{v_i v_j \in \mathcal{E}_2} \\ &\quad + |\mathcal{E}_3| \left(\frac{1}{d_{v_i} \cdot d_{v_j}} \right)_{v_i v_j \in \mathcal{E}_3} + |\mathcal{E}_4| \left(\frac{1}{d_{v_i} \cdot d_{v_j}} \right)_{v_i v_j \in \mathcal{E}_4} \\ &= \frac{1}{2}(p^2 - 3p + 2) \left(\frac{1}{(p^3 - 2)(p^3 - 2)} \right) \\ &\quad + (p^3 - 2p^2 + p) \left(\frac{1}{(p^3 - 2)(p^2 - 2)} \right) \\ &\quad + (p^4 - 2p^3 + p^2) \left(\frac{1}{(p^3 - 2)(p - 1)} \right) \\ &\quad + \frac{1}{2}(p^4 - 2p^3 + p) \left(\frac{1}{(p^2 - 2)(p^2 - 2)} \right) \\ &= \frac{(p-1)(3p^9 - p^8 - 7p^7 - 10p^6 + 9p^5 + 18p^4 + 4p^3 - 4p^2 - 8p - 8)}{2(p^2 - 2)^2(p^3 - 2)^2}. \end{aligned}$$

The general Randić index, with $\alpha \in \mathbb{R}^+$ of $\Gamma(\mathcal{R})$ is given by

$$\begin{aligned} R_\alpha(\Gamma(\mathcal{R})) &= \sum_{v_i v_j \in \mathcal{E}(G)} \left((d_{v_i} \cdot d_{v_j})^\alpha \right) \\ &= |\mathcal{E}_1| \left((d_{v_i} \cdot d_{v_j})^\alpha \right)_{v_i v_j \in \mathcal{E}_1} + |\mathcal{E}_2| \left((d_{v_i} \cdot d_{v_j})^\alpha \right)_{v_i v_j \in \mathcal{E}_2} \\ &\quad + |\mathcal{E}_3| \left((d_{v_i} \cdot d_{v_j})^\alpha \right)_{v_i v_j \in \mathcal{E}_3} \\ &\quad + |\mathcal{E}_4| \left((d_{v_i} \cdot d_{v_j})^\alpha \right)_{v_i v_j \in \mathcal{E}_4} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}(p^2 - 3p + 2)((p^3 - 2)(p^3 - 2))^\alpha + (p^3 - 2p^2 + p)((p^3 - 2)(p^2 - 2))^\alpha \\
&\quad + (p^4 - 2p^3 + p^2)((p^3 - 2)(p - 1))^\alpha + \frac{1}{2}(p^4 - 2p^3 + p)((p^2 - 2)(p^2 - 2))^\alpha \\
&= \frac{1}{2}(p - 1) [(p - 2)(p^3 - 2)^{2\alpha} + p(p^2 - p - 1)(p^2 - 2)^{2\alpha}] \\
&\quad + p(p - 1)^2(p^3 - 2)^\alpha [(p^2 - 2)^\alpha + p(p - 1)^\alpha].
\end{aligned}$$

The inverse general Randić index, with $\alpha \in \mathbb{R}^+$ of $\Gamma(\mathcal{R})$ is given by

$$\begin{aligned}
RR_\alpha(\Gamma(\mathcal{R})) &= \sum_{v_i v_j \in \mathcal{E}(G)} \left(\frac{1}{(d_{v_i} \cdot d_{v_j})^\alpha} \right) \\
&= |\mathcal{E}_1| \left(\frac{1}{(d_{v_i} \cdot d_{v_j})^\alpha} \right)_{v_i v_j \in \mathcal{E}_1} + |\mathcal{E}_2| \left(\frac{1}{(d_{v_i} \cdot d_{v_j})^\alpha} \right)_{v_i v_j \in \mathcal{E}_2} \\
&\quad + |\mathcal{E}_3| \left(\frac{1}{(d_{v_i} \cdot d_{v_j})^\alpha} \right)_{v_i v_j \in \mathcal{E}_3} + |\mathcal{E}_4| \left(\frac{1}{(d_{v_i} \cdot d_{v_j})^\alpha} \right)_{v_i v_j \in \mathcal{E}_4} \\
&= \frac{1}{2}(p^2 - 3p + 2) \left(\frac{1}{((p^3 - 2)(p^3 - 2))^\alpha} \right) \\
&\quad + (p^3 - 2p^2 + p) \left(\frac{1}{((p^3 - 2)(p^2 - 2))^\alpha} \right) \\
&\quad + (p^4 - 2p^3 + p^2) \left(\frac{1}{((p^3 - 2)(p - 1))^\alpha} \right) \\
&\quad + \frac{1}{2}(p^4 - 2p^3 + p) \left(\frac{1}{((p^2 - 2)(p^2 - 2))^\alpha} \right) \\
&= \frac{1}{2} \left(\frac{p - 2}{(p^3 - 2)^{2\alpha}} + \frac{p(p^2 - p - 1)}{(p^2 - 2)^{2\alpha}} \right) \\
&\quad + \frac{p(p - 1)^2}{(p^3 - 2)^\alpha} \left(\frac{1}{(p^2 - 2)^\alpha} + \frac{p}{(p - 1)^\alpha} \right).
\end{aligned}$$

Theorem 4.3. If $\mathcal{R} = \mathbb{Z}_p[x]/\langle x^4 \rangle$, with any prime p , then

$$SSD_3(\Gamma(\mathcal{R})) = 3p^{11} - 6p^{10} + p^9 - 24p^8 + 61p^7 - 10p^6 - 15p^5 - 80p^4 + 68p^3 + 18p^2 - 16$$

$$SSD_5(\Gamma(\mathcal{R})) = \frac{(p-1)(p^{10} - 6p^7 - p^5 + 5p^4 + 16p^3 - 2p^2 - 8p - 8)}{(p^2 - 2)(p^3 - 2)}.$$

Proof. By **Lemma 4.1**, we have the third and fifth symmetric division of $\Gamma(\mathcal{R})$ respectively is

$$\begin{aligned}
SSD_3(\Gamma(\mathcal{R})) &= \sum_{v_i v_j \in \mathcal{E}(G)} (d_{v_i} \cdot d_{v_j} (d_{v_i} + d_{v_j})) \\
&= |\mathcal{E}_1| (d_{v_i} \cdot d_{v_j} (d_{v_i} + d_{v_j}))_{v_i v_j \in \mathcal{E}_1} \\
&\quad + |\mathcal{E}_2| (d_{v_i} \cdot d_{v_j} (d_{v_i} + d_{v_j}))_{v_i v_j \in \mathcal{E}_2} \\
&\quad + |\mathcal{E}_3| (d_{v_i} \cdot d_{v_j} (d_{v_i} + d_{v_j}))_{v_i v_j \in \mathcal{E}_3}
\end{aligned}$$

$$\begin{aligned}
&+ |\mathcal{E}_4| (d_{v_i} \cdot d_{v_j} (d_{v_i} + d_{v_j}))_{v_i v_j \in \mathcal{E}_4} \\
&= \frac{1}{2}(p^2 - 3p + 2)((p^3 - 2)^2(2p^3 - 4)) \\
&\quad + (p^3 - 2p^2 + p)((p^3 - 2)(p^2 - 2)(p^3 + p^2 - 4)) \\
&\quad + (p^4 - 2p^3 + p^2)((p^3 - 2)(p - 1)(p^3 + p - 3)) \\
&\quad + \frac{1}{2}(p^4 - 2p^3 + p)((p^2 - 2)^2(2p^2 - 4)) \\
&= 3p^{11} - 6p^{10} + p^9 - 24p^8 + 61p^7 - 10p^6 \\
&\quad - 15p^5 - 80p^4 + 68p^3 + 18p^2 - 16.
\end{aligned}$$

$$\begin{aligned}
SSD_5(\Gamma(\mathcal{R})) &= \sum_{v_i v_j \in \mathcal{E}(G)} \left(\frac{d_{v_i}}{d_{v_j}} + \frac{d_{v_j}}{d_{v_i}} \right) \\
&= |\mathcal{E}_1| \left(\frac{d_{v_i}}{d_{v_j}} + \frac{d_{v_j}}{d_{v_i}} \right)_{v_i v_j \in \mathcal{E}_1} + |\mathcal{E}_2| \left(\frac{d_{v_i}}{d_{v_j}} + \frac{d_{v_j}}{d_{v_i}} \right)_{v_i v_j \in \mathcal{E}_2} \\
&\quad + |\mathcal{E}_3| \left(\frac{d_{v_i}}{d_{v_j}} + \frac{d_{v_j}}{d_{v_i}} \right)_{v_i v_j \in \mathcal{E}_3} + |\mathcal{E}_4| \left(\frac{d_{v_i}}{d_{v_j}} + \frac{d_{v_j}}{d_{v_i}} \right)_{v_i v_j \in \mathcal{E}_4} \\
&= \frac{1}{2}(p^2 - 3p + 2) \left(\frac{p^3 - 2}{p^3 - 2} + \frac{p^3 - 2}{p^3 - 2} \right) \\
&\quad + (p^3 - 2p^2 + p) \left(\frac{p^3 - 2}{p^2 - 2} + \frac{p^2 - 2}{p^3 - 2} \right) \\
&\quad + (p^4 - 2p^3 + p^2) \left(\frac{p^3 - 2}{p - 1} + \frac{p - 1}{p^3 - 2} \right) \\
&\quad + \frac{1}{2}(p^4 - 2p^3 + p) \left(\frac{p^2 - 2}{p^2 - 2} + \frac{p^2 - 2}{p^2 - 2} \right) \\
&= \frac{p - 1}{(p^2 - 2)(p^3 - 2)} \\
&\quad \times (p^{10} - 6p^7 - p^5 + 5p^4 + 16p^3 - 2p^2 - 8p - 8).
\end{aligned}$$

Theorem 4.4. If $\mathcal{R} = \mathbb{Z}_p[x]/\langle x^4 \rangle$, with any prime p , then

$$\begin{aligned}
H(\Gamma(\mathcal{R})) &= \frac{(p - 1)}{2(p^3 - 2)(p^3 + p^2 - 4)(p^3 + p - 3)(p^2 - 2)} \\
&\quad \times (p^{12} + 4p^{11} + 3p^{10} - 25p^9 - 27p^8 + 12p^7 \\
&\quad + 97p^6 + 39p^5 - 60p^4 - 108p^3 - 12p^2 + 32p + 48)
\end{aligned}$$

$$\begin{aligned}
I(\Gamma(\mathcal{R})) &= \frac{(p - 1)}{4(p^3 + p^2 - 4)(p^3 + p - 3)} \\
&\quad \times (p^{11} + 9p^{10} - 12p^9 - 17p^8 - 42p^7 + 84p^6 + 51p^5 \\
&\quad + 34p^4 - 164p^3 - 20p^2 + 32p + 48)
\end{aligned}$$

$$F(\Gamma(\mathcal{R})) = p^{10} - p^9 + p^8 - 7p^7 + p^6 + 2p^5 + 30p^4 - 28p^3 - 7p^2 + 8.$$

Proof. By applying **Lemma 4.1**, we obtain the Harmonic index

of $\Gamma(\mathcal{R})$ as

$$\begin{aligned}
 H(\Gamma(\mathcal{R})) &= \sum_{v_i v_j \in \mathcal{E}(G)} \left(\frac{2}{d_{v_i} + d_{v_j}} \right) \\
 &= |\mathcal{E}_1| \left(\frac{2}{d_{v_i} + d_{v_j}} \right)_{v_i v_j \in \mathcal{E}_1} + |\mathcal{E}_2| \left(\frac{2}{d_{v_i} + d_{v_j}} \right)_{v_i v_j \in \mathcal{E}_2} \\
 &\quad + |\mathcal{E}_3| \left(\frac{2}{d_{v_i} + d_{v_j}} \right)_{v_i v_j \in \mathcal{E}_3} + |\mathcal{E}_4| \left(\frac{2}{d_{v_i} + d_{v_j}} \right)_{v_i v_j \in \mathcal{E}_4} \\
 &= \frac{1}{2} (p^2 - 3p + 2) \left(\frac{2}{(p^3 - 2) + (p^3 - 2)} \right) \\
 &\quad + (p^3 - 2p^2 + p) \left(\frac{2}{(p^3 - 2) + (p^2 - 2)} \right) \\
 &\quad + (p^4 - 2p^3 + p^2) \left(\frac{2}{(p^3 - 2) + (p - 1)} \right) \\
 &\quad + \frac{1}{2} (p^4 - 2p^3 + p) \left(\frac{2}{(p^2 - 2) + (p^2 - 2)} \right) \\
 &= \frac{(p - 1)}{2(p^3 - 2)(p^3 + p^2 - 4)(p^3 + p - 3)(p^2 - 2)} \\
 &\quad \times (p^{12} + 4p^{11} + 3p^{10} - 25p^9 - 27p^8 + 12p^7 \\
 &\quad + 97p^6 + 39p^5 - 60p^4 - 108p^3 - 12p^2 + 32p \\
 &\quad + 48).
 \end{aligned}$$

The inverse sum index of $\Gamma(\mathcal{R})$ is

$$\begin{aligned}
 I(\Gamma(\mathcal{R})) &= \sum_{v_i v_j \in \mathcal{E}(G)} \left(\frac{d_{v_i} \cdot d_{v_j}}{d_{v_i} + d_{v_j}} \right) \\
 &= |\mathcal{E}_1| \left(\frac{d_{v_i} \cdot d_{v_j}}{d_{v_i} + d_{v_j}} \right)_{v_i v_j \in \mathcal{E}_1} + |\mathcal{E}_2| \left(\frac{d_{v_i} \cdot d_{v_j}}{d_{v_i} + d_{v_j}} \right)_{v_i v_j \in \mathcal{E}_2} \\
 &\quad + |\mathcal{E}_3| \left(\frac{d_{v_i} \cdot d_{v_j}}{d_{v_i} + d_{v_j}} \right)_{v_i v_j \in \mathcal{E}_3} + |\mathcal{E}_4| \left(\frac{d_{v_i} \cdot d_{v_j}}{d_{v_i} + d_{v_j}} \right)_{v_i v_j \in \mathcal{E}_4} \\
 &= \frac{1}{2} (p^2 - 3p + 2) \left(\frac{(p^3 - 2)(p^3 - 2)}{(p^3 - 2) + (p^3 - 2)} \right) \\
 &\quad + (p^3 - 2p^2 + p) \left(\frac{(p^3 - 2)(p^2 - 2)}{(p^3 - 2) + (p^2 - 2)} \right) \\
 &\quad + (p^4 - 2p^3 + p^2) \left(\frac{(p^3 - 2)(p - 1)}{(p^3 - 2) + (p - 1)} \right) \\
 &\quad + \frac{1}{2} (p^4 - 2p^3 + p) \left(\frac{(p^2 - 2)(p^2 - 2)}{(p^2 - 2) + (p^2 - 2)} \right) \\
 &= \frac{(p - 1)}{4(p^3 + p^2 - 4)(p^3 + p - 3)} \\
 &\quad \times (p^{11} + 9p^{10} - 12p^9 - 17p^8 - 42p^7 \\
 &\quad + 84p^6 + 51p^5 + 34p^4 - 164p^3 \\
 &\quad - 20p^2 + 32p + 48).
 \end{aligned}$$

The Forgotten index of $\Gamma(\mathcal{R})$ is

$$\begin{aligned}
 F(\Gamma(\mathcal{R})) &= \sum_{v_i v_j \in \mathcal{E}(G)} (d_{v_i}^2 + d_{v_j}^2) \\
 &= |\mathcal{E}_1| (d_{v_i}^2 + d_{v_j}^2)_{v_i v_j \in \mathcal{E}_1} + |\mathcal{E}_2| (d_{v_i}^2 + d_{v_j}^2)_{v_i v_j \in \mathcal{E}_2} \\
 &\quad + |\mathcal{E}_3| (d_{v_i}^2 + d_{v_j}^2)_{v_i v_j \in \mathcal{E}_3} + |\mathcal{E}_4| (d_{v_i}^2 + d_{v_j}^2)_{v_i v_j \in \mathcal{E}_4} \\
 &= \frac{1}{2} (p^2 - 3p + 2) ((p^3 - 2)^2 + (p^3 - 2)^2) \\
 &\quad + (p^3 - 2p^2 + p) ((p^3 - 2)^2 + (p^2 - 2)^2) \\
 &\quad + (p^4 - 2p^3 + p^2) ((p^3 - 2)^2 + (p - 1)^2) \\
 &\quad + \frac{1}{2} (p^4 - 2p^3 + p) ((p^2 - 2)^2 + (p^2 - 2)^2) \\
 &= p^{10} - p^9 + p^8 - 7p^7 + p^6 + 2p^5 + 30p^4 - \\
 &\quad 28p^3 - 7p^2 + 8.
 \end{aligned}$$

Following the analytical formulas of several degree-based indices in the previous result, we conducted a numerical comparison of these indices of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$. For both the general Randić index R_α and its inverse RR_α , we set $\alpha = 1$. The corresponding results are graphically represented in 2, with a summary provided in Table 2.

Based on the numerical results in Table 2 and the plots in Figure 2, the topological indices of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$ grow at very different rates. Among them, the symmetric division index SSD_3 produces the highest values as p increases. This shows that SSD_3 is highly sensitive to the growth of the graph and emphasizes the influence of high-degree vertices. The Forgotten index F and the general Randić index R_1 also take very large values, although still much smaller than SSD_3 . Their numerical trends are quite similar, which indicates that both indices respond in a comparable way to the distribution of vertex degrees. The inverse sum index I and the fifth symmetric division index SSD_5 give values that are in the middle range: not as small as RR_1 , mM_2 , or H , but also not as large as F , R_1 , or SSD_3 . In contrast, the inverse Randić index RR_1 , the modified Zagreb index mM_2 , and the harmonic index H remain consistently small for all primes p . These indices mainly reflect local structural features or normalized interactions between vertex degrees, so their values do not increase drastically as the graph becomes larger. The overall order from smallest to largest is as follows:

$$\begin{aligned}
 RR_1(\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)) &< {}^mM_2(\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)) \\
 &< H(\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)) \\
 &< I(\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)) \\
 &< SSD_5(\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)) \\
 &< R_1(\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)) \\
 &< F(\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)) \\
 &< SSD_3(\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)).
 \end{aligned}$$

Table 2. Values of Various Topological Indices for $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$ for Several Primes p

p	mM_2	R_1	RR_1	SSD_3	SSD_5	H	I	F
2	1.0833	52	1.0833	376	33.333	2.1429	7.4286	236
3	1.0963	5260	0.94243	1.5734×10^5	531.1	5.5995	197.29	33452
5	1.2009	6.144×10^5	0.93118	8.499×10^7	13148	15.705	5653.7	7.6932×10^6
7	1.2678	1.1294×10^7	0.94288	4.1878×10^9	1.0390×10^5	29.829	4.3601×10^4	2.4233×10^8
11	1.3408	4.9915×10^8	0.95975	6.9869×10^{11}	1.6326×10^6	70.005	6.2684×10^5	2.366×10^{10}
13	1.3625	1.9824×10^9	0.96515	4.5441×10^{12}	4.5007×10^6	96.064	1.6629×10^6	1.2764×10^{11}
17	1.3921	1.7877×10^{10}	0.97261	9.0696×10^{13}	2.2857×10^7	160.15	7.9435×10^6	1.9015×10^{12}
19	1.4026	4.4299×10^{10}	0.97528	3.1265×10^{14}	4.4792×10^7	198.18	1.5191×10^7	5.8192×10^{12}
23	1.4185	2.0956×10^{11}	0.97933	2.61×10^{15}	1.4209×10^8	286.23	4.6317×10^7	3.968×10^{13}
29	1.4345	1.372×10^{12}	0.98341	3.4081×10^{16}	5.7559×10^8	448.28	1.7962×10^8	4.0658×10^{14}
31	1.4385	2.3533×10^{12}	0.98444	7.1315×10^{16}	8.6055×10^8	510.29	2.6546×10^8	7.9385×10^{14}
37	1.4481	9.8281×10^{12}	0.98688	5.0495×10^{17}	2.4998×10^9	720.32	7.495×10^8	4.6815×10^{15}

4. CONCLUSIONS

The present study revisits the spectral properties of the zero-divisor graph of the ring $\mathbb{Z}_p[x]/\langle x^4 \rangle$, addressing inaccuracies in earlier present in the lower bound for graph energy proposed by Johnson and Sankar (2023). While recent literature by Rather resolved these discrepancies using vertex partitioning and a quotient matrix approach, our work provides an alternative framework via Schur complement-based reduction to derive the characteristic polynomial and its corresponding eigenvalues. This distinct matrix-theoretic approach independently confirms and reinforces the corrected energy bounds, both lower and upper, obtained by Rather. Furthermore, this study extends the existing literature beyond the scope of both Johnson & Sankar and Rather by conducting an extensive investigation into various degree-based topological indices that remained previously unexplored. The degree-based topological indices increase with the prime p , with several indices exhibiting polynomial growth as the size and degree distribution of $\Gamma(\mathbb{Z}_p[x]/\langle x^4 \rangle)$ increase. Although the magnitudes of the indices differ considerably, their growth patterns are consistent with the underlying graph structure and the derived closed-form expressions.

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